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ARTICLES

The Region Unknotting Game

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Unknotting a piece of rope may be a very tricky task. But if the rope consists of a single strand with two free ends, unknotting is always possible in theory, even if it is a frustrating experience in practice. We can always "slide the knot" off the end of the rope or "unwind" the knot in order to produce an unknotted piece of rope. This is why, when mathematicians consider knots, we typically consider knots that occur in circles. So if a nontrivial knot occurs, there is no way to remove it by sliding the knot off the free end. Indeed, knots in circles can get quite complicated with no hope of simplification. That is, there is no hope *unless* we are prepared to change the structure of the knot by cutting and pasting the knot at various points. Such moves that allow us to simplify the topological knotting in a knot are called **unknotting operations**.

The most fundamental example of an unknotting operation is a **crossing change**. A crossing change can be performed on a *knot diagram* (in other words, a nice 2-dimensional picture of a knot), and it usually changes the *topological type* of the knot (unless the crossing is *cosmetic* or *nugatory* as in [**9**]). We see this crossing change operation pictured in Figure 1.

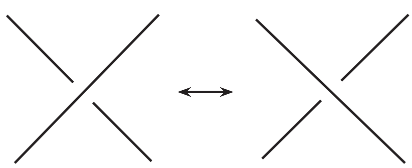

Figure 1 A crossing change.

An interesting fact about knots that a young knot theorist might prove early on in her investigation is the following. Given any diagram of a knot, it is possible to do some number of crossing changes to produce a diagram of the unknot (that is, the unknotted circle). Indeed, we can take any knot diagram and, by changing certain crossings, turn

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it into an unknotted *descending* diagram. In Figure 2, we see a knot diagram and a descending diagram that can be obtained from it by traveling around the knot from a starting point and performing crossing changes when necessary to guarantee that each time we encounter a crossing for the first time, we pass *over* it. We can convince ourselves that a descending diagram is actually unknotted with the following thought experiment. Imagine picking up the knot at the point where we started traveling around the knot. If we lift the knot from here, the strands of the knot below will simply unravel to form an unknotted circle. We can try this thought experiment on the diagram on the right in Figure 2.

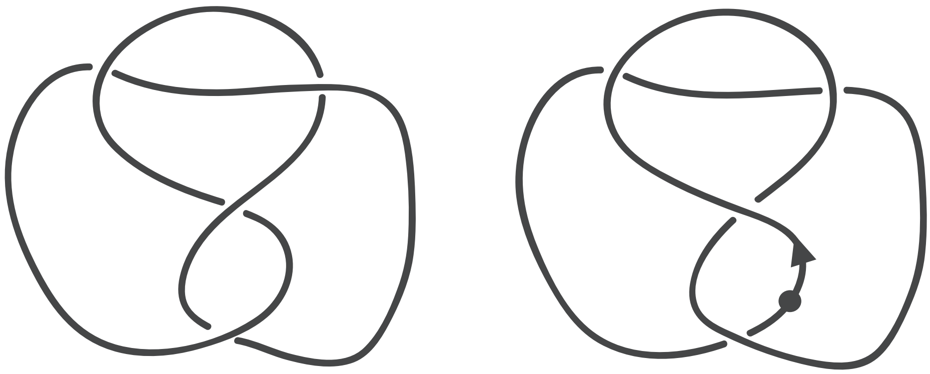

Figure 2 A knot diagram of one of its descending relatives.

There are several, generally more complicated operations that can be performed on knot diagrams to produce diagrams of the unknot, for instance the # and Δ moves [**5, 6**]. The focus of our work will be on an unknotting operation called the **region crossing change**, or RCC move, that was only recently discovered by Ayaka Shimizu [**8**]. Here's how it works. To perform a region crossing change on a knot diagram, we first select a region R. Then we perform the crossing change operation on all crossings along the boundary of R. An example is pictured in Figure 3.

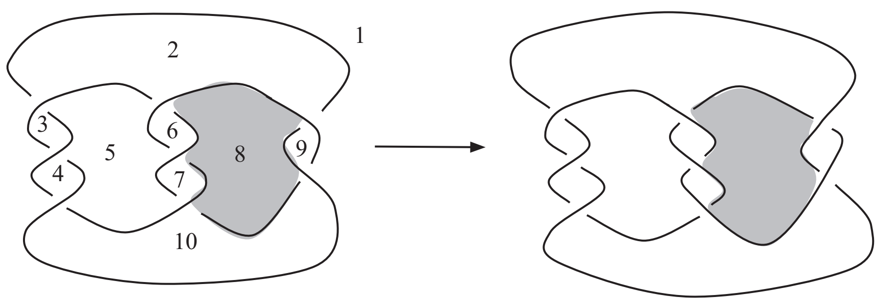

Figure 3 The region crossing change operation applied to region **8**. What would the diagram look like if we had performed an RCC move on region 1 or region 2 instead?

While it is fairly easy to convince ourselves that individual crossing changes can unknot knots, it is really quite surprising that region crossing changes can do the same. An enlightening proof of this fact can be found in [**8**].

Fun & Games We digress to talk about our other source of inspiration: games on knot diagrams. In [**3**], several knot games were described for the first time. This prompted further study of knot games in [**2, 4**]. The principal knot game that was studied in [**3**] and [**4**] came to be referred to as the **Knotting–Unknotting Game**.

The Knotting–Unknotting Game can be played as follows. Two players are given a knot diagram. One player, U, wants to make this diagram into the diagram of an unknot, while the other player, K, wants to make the diagram into a diagram of a nontrivial knot. During game play, U and K take turns choosing crossings. When a crossing is chosen by a player, the player can decide to (a) keep it as is and fix the crossing information for the rest of the game, or (b) change the crossing and then fix its crossing information for the remainder of the game. In either case, when a player's turn is over, the crossing they have just played on is removed from game play. The game continues until all crossings have been played upon, and the knot type of the final diagram determines the game's winner. We give one example of game play here in Figure 4 to give the reader an idea of how the game works. To see other examples of this game being played, we recommend reading [**1**] and [**3**].

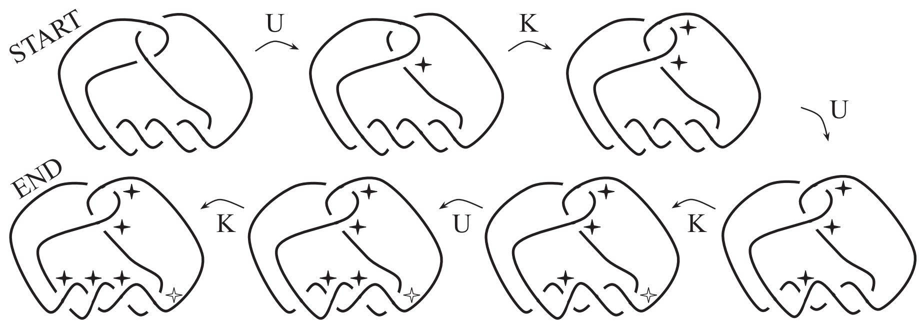

Figure 4 The Knotting–Unknotting Game is played on a type of knot called a twist knot. U plays first. Four-pointed stars indicate which crossings have been played upon during the course of the game. Specifically, a black star indicates that a crossing change was performed while a white star indicates that a crossing was fixed without being changed. Notice that a crossing is changed on each turn except for K's second turn. The final diagram is a complicated diagram of a figure eight knot (also pictured on the left in Figure 2). Since this is a nontrivial knot, K has won the game!

The fact that the Knotting–Unknotting Game uses crossing changes gave us the following idea. What if we played a variation on this game that uses *region* crossing changes instead of ordinary crossing changes? There could still be two players, U and K, with the same unknotting and knotting goals, and these players could still take turns making moves on a knot diagram. But perhaps instead of performing crossing changes and fixing crossings, the players could perform region crossing changes and, in a sense, "fix" regions in the plane by removing them from game play. We call this new game the **Region Unknotting Game**. This game is not to be confused with Region Select, a delightful *single-player* game invented by Akio Kawauchi, Ayaka Shimizu, and Kengo Kishimoto that also uses the region crossing change operation [**7**].

We give a sample game in Figure 5. Notice that in this example, the diagram becomes unknotted after U's third move. Fortunately for K, he is able to immediately return the diagram to a knotted state and win the game.

Now that we know how to play the game, there are some obvious questions for us to ask. Given a specific knot diagram, which player has an advantage if the unknotter plays first? What if the knotter plays first? Does who plays first change who has a winning strategy? For those unfamiliar with combinatorial game theory, let us be more precise. A **winning strategy** in a game is an optimal strategy that guarantees that a given player will win the game, regardless of how their opponent plays. In general,

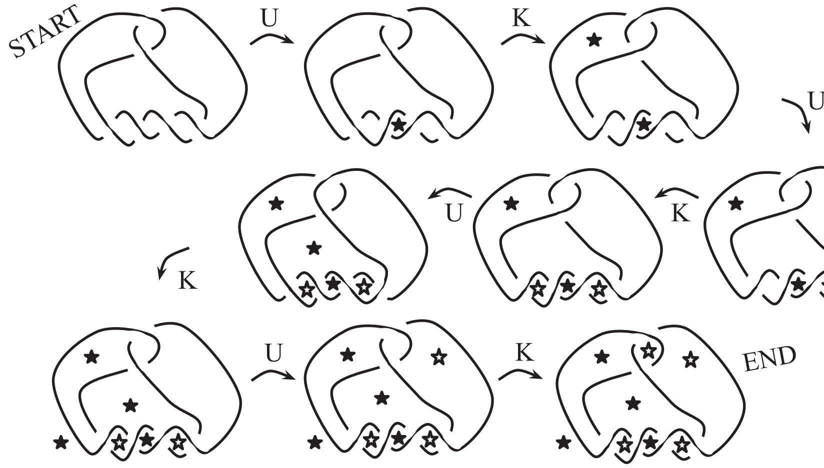

Figure 5 The Region Unknotting Game is played on a twist knot. The unknotter, U, plays first. Stars indicate which regions have been played upon during the course of the game: a solid star indicates that a region crossing change was performed while a hollow star indicates that a region was played without being changed. Notice that the outer region was played upon—as indicated by a star outside of the knot diagram—and this was a key move for the knotter. The final diagram is a complicated diagram of the trefoil, the smallest nontrivial knot. (See if you can use your spatial intuition to redraw this knot with only three crossings.) Thus, K has won again!

one or more moves that follow a winning strategy may seem counterintuitive, given the player's goal. However, these moves will guarantee a win by the end of game play.

In the next sections, we will answer some of our game-theoretical questions for specific knot diagrams. We will also propose some simple questions for which we don't have the answers.

Twisting and untwisting twist knots

Definitions and tools In Figure 5, we saw a sample Region Unknotting Game played on a diagram of a type of knot called a **twist knot**. This family of knots is actually a nice place to begin our investigation, partly because any of these minimal crossing diagrams can be unknotted with a single region crossing change. We recommend staring at the examples of twist knot diagrams in Figure 6 to convince yourselves of this fact.

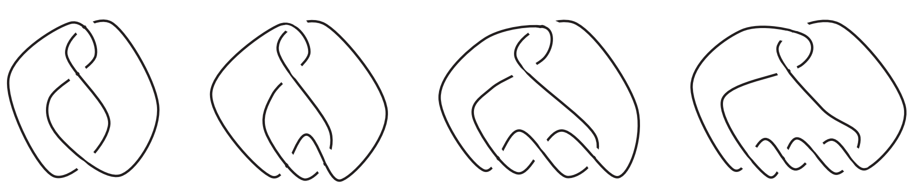

Figure 6 Several members of the twist knot family.

We would like to know who has a winning strategy in the Region Unknotting Game on standard, *alternating* diagrams of twist knots, as in Figure 6. (Just as you might have guessed, we say a knot diagram is **alternating** if, as we travel around the diagram, we pass through crossings following an over, under, over, under pattern.)

So let us ask the following questions:

1. For standard twist knot diagrams, does which player goes first affect who has a winning strategy?
2. Does the parity of the number of crossings in the twist knot diagram affect who has a winning strategy?

Before we begin our investigation, we introduce a few more notions. The first is the notion of the **sign** of a crossing. To determine the sign of a crossing in a knot diagram, we first give the diagram an *orientation*. That is, we'll choose a direction to travel around the knot. It does not matter which direction we choose, but we must make a choice. Once we have an orientation, we look at the relationship between the orientations of strands involved in a crossing and which strand is over/under. We assign a crossing a **positive** sign or a **negative** sign following the convention shown in Figure 7. Notice that, if we view a crossing from the perspective that both strands involved in the crossing are oriented upwards, then the crossing is positive if the overstrand is oriented up and to the right and negative if the overstrand is oriented up and to the left. The sign of the crossing is sometimes also referred to as the **local writhe**.

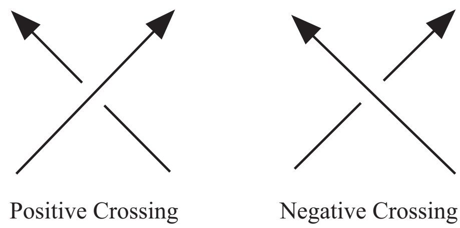

Figure 7 The sign of a crossing.

How can we see that the sign of a crossing in a knot diagram does not actually depend on the particular orientation chosen? In Figure 8, we give a diagram of a trefoil with both possible orientations to illustrate why this works. Notice that all three crossings are negative in both diagrams. (Dear reader, why don't the signs of the crossings change?)

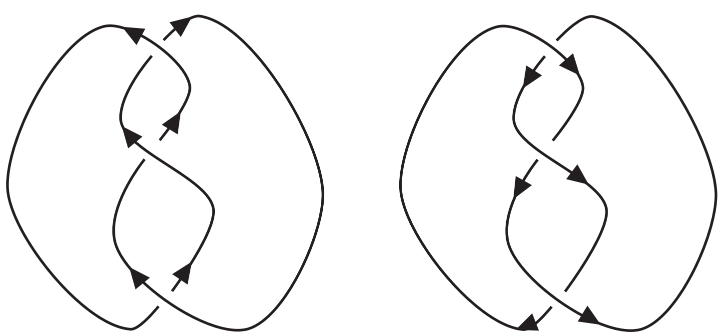

Figure 8 A knot diagram with three negative crossings, shown with two different orientations. Turn your head upside down to convince yourself that all three crossings are negative in the diagram on the right.

Next, we describe a local move that can be performed on a knot diagram without changing the topological type of the knot. Named after its discoverer, Kurt Reidemeister, this move is usually referred to as **Reidemeister 2**, or **R2** for short. The R2 move, which is pictured in Figure 9, simply slides one strand of the knot over another, creating or removing two crossings. Being able to picture instances of this move will help us visualize the simplification of various knot diagrams in the games we will consider.

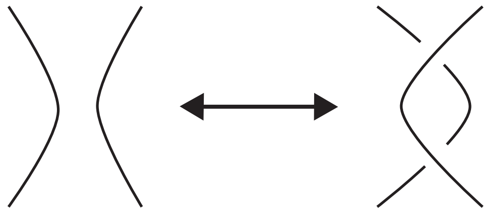

Figure 9 The Reidemeister 2 move.

The simplest twist knots Now, without further ado, let us take a look at which strategies can be used to guarantee a win. We'd like to explore winning strategies for *all* twist knots, but we will start with the smallest odd and even cases to develop some intuition first. We observe that the simplest odd twist knot diagram is a diagram of the simplest nontrivial knot, the trefoil. The simplest even twist knot diagram is a diagram of the nontrivial knot called figure eight knot that we first saw in Figure 2. These two knots will provide our starting game boards.

Proposition 1. *If the Region Unknotting Game is played on the twist knot diagram with three crossings pictured in Figure 10, then the unknotter has a winning strategy.*

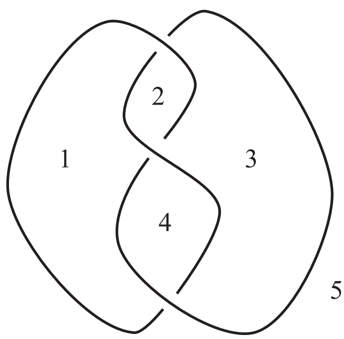

Figure 10 Twist knot with three negative crossings.

We'll prove this proposition in two lemmas. In the first, Lemma 1, we consider the case where the unknotter plays first. In the second, Lemma 2, we investigate how the unknotter can win if she plays second.

Lemma 1. *If the Region Unknotting Game is played on the twist knot diagram with three crossings pictured in Figure 10 and the unknotter plays first, then the unknotter has a winning strategy.*

Proof. First, we observe two facts: (i) all crossings at the start of the game are negative, and (ii) the knotter needs to ensure that all the crossings have the same sign in order to win. If all crossings are positive or all crossings are negative, then the diagram represents the trefoil knot. Any other combination of signs for the three crossings corresponds to a diagram of the unknot. (The reader should take a moment to be convinced of this fact before proceeding.) Thus, the unknotter's goal is to change one or two, but not all three, of the crossings to the opposite sign by the end of the game. In order to achieve this, the unknotter must ensure that either one or two, but not all three, of the regions 2, 4, and 5 have been changed, since these are precisely the regions that affect exactly two of the crossings.

We observe that regions 1 and 3 do not affect the overall knotting or unknotting of the diagram since they both affect all three crossings. Since the unknotter plays first and there are an odd number of regions, the unknotter need not move in the pair of regions $\{1, 3\}$ first. When the knotter moves on one of the regions in the pair, the unknotter can respond by moving on the other region (say, keeping the region the same). Thus, we may safely focus on the remaining three regions: 2, 4, and 5.

The unknotter has the first move in the set $\{2, 4, 5\}$ of regions. She should perform a region crossing change on one of these regions. Without loss of generality, assume the unknotter changes region 2. This has the effect of producing two same-sign crossings and one opposite-sign crossing. We now have the following possibilities.

- If the knotter keeps region 4 (resp. 5) the same, preserving the crossing signs, then the unknotter wins by keeping region 5 (resp. 4) the same.
- If the knotter changes region 4 (resp. 5), then the unknotter still wins by keeping region 5 (resp. 4) the same.

The net effect in either case is that either one or two but not all three of regions 2, 4, and 5 have been changed so that, by the end of the game, two crossings have the same sign and the remaining crossing has the opposite sign. Thus, the unknotter wins when playing first. ■

Lemma 2. *If the Region Unknotting Game is played on the twist knot diagram with three crossings pictured in Figure 10 and the unknotter plays second, then the unknotter has a winning strategy.*

Proof. Let us consider the case where the knotter plays first on the twist knot diagram with three crossings. We can safely ignore regions 1 and 3, since they do not effect the knotting of the diagram, and the unknotter can guarantee she will play on exactly one of the regions. Instead, we focus on the regions in the set $\{2, 4, 5\}$. There are two subcases to consider.

- If the knotter keeps one of regions 2, 4, and 5 the same, then the crossings still all have the same sign. The unknotter can guarantee a win by changing one of the two remaining regions, which ensures that either one or two of regions 2, 4, and 5 will have been changed at the end of game play. The result is that exactly two of the three crossings will have the same sign when the game is over.
- If the knotter changes one of regions 2, 4, and 5, then exactly two of the crossing will have the same sign. In this case, the unknotter should keep another region the same, preserving the signs of all crossings. Regardless of whether the knotter changes or preserves the remaining region, exactly two crossings will have the same sign when the game is over.

Thus, the unknotter also wins when playing second. ■

Let us move on and consider the twist knot diagram with the smallest even number of crossings.

Proposition 2. *If the Region Unknotting Game is played on the twist knot diagram with four crossings in Figure 11, then the second player has the winning strategy, regardless of which player plays second.*

Proof. In the knot diagram pictured in Figure 11, each region has a sister region that has a similar effect on the game. These region pairs are: $\{1, 3\}$, $\{2, 5\}$, and $\{4, 6\}$. If

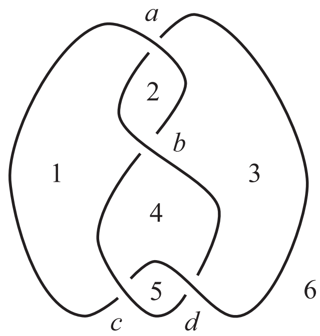

Figure 11 Twist knot with four crossings: a and b are positive and c and d are negative.

we study the diagram, we can see that regions 2 and 5 each affect exactly two adjacent crossings, regions 1 and 3 both effect crossings a and b and exactly one of the crossings in the pair $\{c, d\}$, and regions 4 and 6 both affect crossings c and d and exactly one of the crossings in the pair $\{a, b\}$. We will exploit this pairing to show that for every move player 1 has, player 2 has a countermove.

Consider the case when the unknotter plays first. We claim that the knotter has a winning strategy. To win the game, the knotter must ensure that crossings a and b have the same sign and crossings c and d have the same sign. Indeed, if both pairs have the same sign, then the resulting diagram is the trefoil (in the case that all four signs are positive or all are negative) or the figure eight knot (if two signs are positive and two are negative, as in Figure 11). The unknotter will either play in the {1, 3} pair, the {2, 5} pair, or the {4, 6} pair. Let us consider each of these options.

- First, notice that regions 2 and 5 do not affect the overall knotting or unknotting of the diagram. Either both crossings surrounding region 2 (resp. 5) are the same sign or opposite signs, and performing an RCC move preserves this property. If the unknotter plays in region 2 (resp. 5), the knotter can play in 5 (resp. 2). Without loss of generality, the knotter can mirror the unknotter's move (so that both players have performed an RCC move on a region or both have chosen regions without performing RCC moves).
- Now suppose that the unknotter plays in region 1 (resp. 3). Then the knotter can play in the region 3 (resp. 1) in such a way that crossings c and d have the same sign after the pair of regions has been played.
- Suppose the unknotter plays in region 4 (resp. 6). Then the knotter can play in region 6 (resp. 4) in such a way that crossings a and b have the same sign after the pair of regions has been played.

The order in which these three plays are made does not affect the outcome of the game. For crossing a, the only region that has an affect on a but not b is region 6. So changing any other region does not affect the property that a and b have the same sign. Similarly, the only region that has an affect on b but not a is region 4. Analogously, the only region that has an affect on c but not d is region 1. Similarly, the only region that has an affect on d but not c is region 3. Once the knotter has guaranteed that a and b have the same sign and c and d have the same sign, the knotter wins when playing second.

Now consider the case when the knotter plays first. To win the game playing second, the unknotter must ensure that crossings a and b have the opposite sign or crossings c and d have the opposite sign so that a simplifying R2 move is possible. (In fact, she will achieve both of these goals, either one of which would guarantee a win for the unknotter.) To be sure, if the unknotter can achieve this goal, the resulting diagram is the unknot. Once again, the knotter will either play in the {1, 3} pair, the {2, 5} pair, or the {4, 6} pair. Let us consider each of these options.

- Recall that regions 2 and 5 do not affect the overall knotting or unknotting of the diagram. If the knotter plays in region 2 (resp. 5), the unknotter can play in 5 (resp. 2), mirroring the knotter's move.
- Suppose the knotter plays in region 1 (resp. 3). Then the unknotter can play in region 3 (resp. 1) in such a way that crossings c and d have opposite signs after the pair of regions has been played.
- Suppose the knotter plays in region 4 (resp. 6). Then the unknotter can play in region 6 (resp. 4) in such a way that crossings a and b have opposite signs after the pair of regions has been played.

Once again, the order in which these three pairs of plays are made does not affect the outcome of the game. So we can be sure that the unknotter will always be able to produce a diagram where a and b have opposite signs and c and d have opposite signs. Thus, the unknotter wins when playing second. ■

General strategies for twisting and untwisting Fortunately, our argument for a twist knot diagram with four crossings can be generalized to provide a winning strategy for the Region Unknotting Game on an alternating twist knot diagram with any *even* number of crossings. In Figure 12, we give a general alternating twist knot diagram.

A single observation holds the key to unlocking winning strategies for the Region Unknotting Game played on this knot diagram. In the figure, crossings a and b have the same sign, so the "clasp" is clasped. If exactly one of those crossings were changed so that crossings a and b had opposite signs, then an R2 move could be used to unravel the entire knot, regardless of the signs of crossings $c_1, c_2, \ldots, c_{n-3}$, and c_{n-2}. The unknotter will use this fact to her advantage as the knotter attempts to defend the clasp!

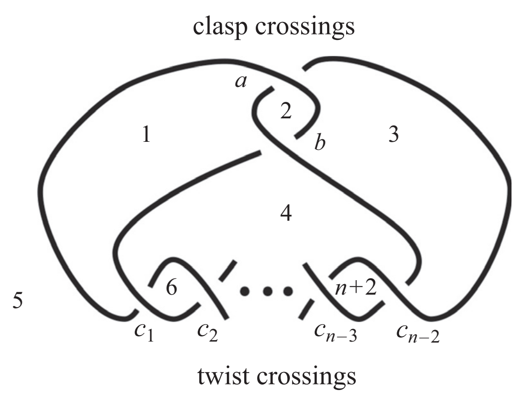

Figure 12 Example of a general twist knot. Crossings a and b are called the *clasp crossings,* while $c_1, c_2, \ldots, c_{n-3}, c_{n-2}$ are called the *twist crossings.*

Theorem 1. *If the Region Unknotting Game is played on a standard alternating twist knot diagram with n crossings where n is even, then the second player has the winning strategy, regardless of which player plays second.*

We prove our theorem in two lemmas. In Lemma 3, we explore the winning strategy for the unknotter playing second, while in Lemma 4, we describe a more complex winning strategy for the knotter playing second.

Lemma 3. *Suppose the Region Unknotting Game is played on a standard alternating twist knot diagram with n crossings, as in Figure 12. If n is even and the unknotter plays second, then the unknotter has a winning strategy.*

Proof. The unknotter, playing second, wins in this game by ensuring that the clasp can be unclasped using an R2 move. She achieves this by ensuring that crossings a and b have opposite signs. Recall that the only region that has an affect on a but not b is region 5, and the only region that affects crossing b but not a is region 4. Changing any other region does not affect whether or not crossings a and b have the same sign. Our strategy requires us to ensure that a and b have opposite signs, so moves on regions other than on 4 or 5 are not relevant for game play. Since the knotter plays first and the knot diagram has an even number of regions, the unknotter will play last. Thus, the unknotter can force the knotter to play first on the pair of regions 4 and 5. This ensures that the unknotter can play on the corresponding region in such a way that crossings a and b have opposite signs at the end of game play. The resulting knot is the unknot; hence, the unknotter wins. ■

Lemma 4. *Suppose the Region Unknotting Game is played on a standard alternating twist knot diagram with n crossings, as in Figure 12. If n is even and the knotter plays second, then the knotter has a winning strategy.*

Proof. Let's begin by getting our bearings. Both clasp crossings, a and b, are positive and the twist crossings are all negative when game play begins.

To determine the knotter's winning strategy, we first observe that the knotter can ensure that the clasp remains clasped (i.e., a and b have the same sign) since the knotter plays second and there are an even number of regions. Just as we noted in Lemma 3, region 4 is the only region that affects crossing b but not a and region 5 is the only region that affects crossing a but not b. The knotter can ensure that the unknotter plays first in the region {4, 5} pair since there are an even number of regions and the knotter plays second. If the unknotter changes region 4 or 5 to create clasp crossings that have opposite signs, then the knotter can change the sister region to return the clasp crossings to a $++$ or $--$ sign configuration. If, on the other hand, the unknotter preserves one of the regions 4 or 5, the knotter simply ought to follow suit by preserving the sister region. Since the unknotter cannot unknot the diagram by undoing the clasp if the knotter follows this strategy, the unknotter's only chance to win is to make the sum of the signs of all the twist crossings equal 0, in which case the entire twist can be untangled using some number of R2 moves.

At this point we make some important observations. First, the sum of the signs of the twist crossings (which we will hereafter refer to as w) is even since there are $n-2$ twist crossings and n is even. Since w is even regardless of what the particular signs of the twist crossings are, it is either congruent to 0 modulo 4 or 2 modulo 4. Second, there are exactly two regions in the diagram that each affect exactly one of the twist crossings, namely regions 1 and 3. Changing *exactly one* of regions 1 and 3 will change whether w is 0 or 2 modulo 4, while changing any of the other regions (namely regions $2, 4, 5, 6, 7, \ldots, n+2$) preserves which congruence class w is in since these regions affect 0, 2, or all of the twist crossings. Thus, when the unknotter plays first on region 1 or 3, the knotter has the opportunity to play on the sister region, ensuring that after regions 1 and 3 are out of game play, w is congruent to 2 modulo 4. After this point, w will remain congruent to 2 modulo 4 for the remainder of the game. So regardless of how our two players play on the remaining regions, w can never equal 0. Hence, the twist can never be completely unknotted and, thus, the final knot diagram is nontrivial. So the knotter has a strategy to win playing second. ■

We've completely characterized the Region Unknotting Game for twist knot diagrams with an even number of crossings, but what if the number of crossings is odd? It turns out that we can find which player has a winning strategy in this situation, *but only when the unknotter plays first*.

Theorem 2. *Suppose the Region Unknotting Game is played on a standard alternating twist knot diagram with n crossings, as in Figure 12. If n is odd and the unknotter plays first, then the unknotter has a winning strategy.*

Proof. Following the same strategy described in Lemma 3, the unknotter wins by ensuring that the clasp is unclasped using an R2 move. Since the unknotter plays first and the knot diagram has an odd number of regions, the unknotter will also play last. Thus, the unknotter can force the knotter to play first on the pair of regions 4 and 5. ■

We have also explored play of the Region Unknotting Game on alternating twist knot diagrams with n crossings, where n is odd and the knotter plays first. However, we were unable to discover a general winning strategy for either the knotter or the unknotter when $n > 3$. In this case, since the knotter plays last, he can ensure that the clasp remains clasped. Thus, a winning strategy for the unknotter must ensure that the sum of the signs of the twist crossings is $w = \pm 1$ so that all but one crossing in the twist can be removed with R2 moves. Moreover, the sum of the signs of *all* crossings in the diagram must be ± 1 for the diagram to be unknotted globally. It is difficult to determine whether the unknotter will be able to accomplish this goal or whether the knotter can always thwart the unknotter's attempts to control the sum of the crossing signs, or the **writhe** of the knot diagram. Who has a winning strategy in this case remains an open question.

Getting a handle on torus knots

What is a torus knot? A **torus knot** is a knot that can be drawn on the surface of a torus (think: doughnut) without creating any crossings on the surface. See Figure 13, and use your imagination to picture how the crossings could be drawn with understrands wrapping around the underside of the torus and overstrands passing on top of the torus.

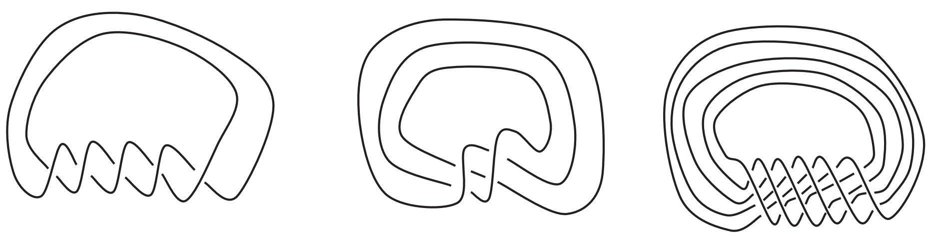

Figure 13 Examples of torus knots. From left to right, we see a $(2, 5)$-torus knot, a $(3, 2)$-torus knot, and a $(5, 6)$-torus knot.

We often refer to torus knots using the terminology (p, q)**-torus knot**. Here p describes the number of times the knot wraps around the longitude of the torus, and q describes the number of times the knot wraps around the meridian of the torus. Notice that in this description, we are not making any distinction between a knot and its mirror image. This is, in part, because the strategies we use for our game are valid both for a given knot and for its mirror image.

Wrapping and unwrapping torus knots In investigating the Region Unknotting Game on torus knot diagrams, we made an interesting observation. We discovered that which diagram of a given torus knot that we use to play our game can have a dramatic effect on the winning strategy. For instance, the (p, q)-torus knot is topologically equivalent to the (q, p)-torus knot, but our standard diagrams for each of these knots are different as are their Region Unknotting Game winning strategies. We first noticed this in the case of the (2, 3) or (3, 2) torus knot, i.e., the trefoil knot. Recall Proposition 1, which states that the unknotter always has a winning strategy on the given diagram of the trefoil. This is not so for the diagram of the trefoil given in Figure 14. In fact, we can prove the following proposition.

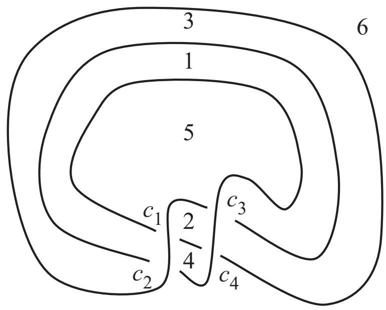

Figure 14 The trefoil, drawn using a standard $(3, 2)$-torus knot diagram.

Proposition 3. *Suppose the Region Unknotting Game is played on the* $(3, 2)$*-torus knot diagram pictured in Figure 14. Then the second player has the winning strategy, regardless of their goal.*

Proof. The diagram is a nontrivial knot if and only if the crossings are homogeneous (i.e., they all have the same sign) or the crossings signs are alternating, i.e.,

$$(c_1, c_2, c_3, c_4) = (+1, -1, +1, -1) \text{ or } (-1, +1, -1, +1).$$

Figure 15 will help you convince yourself of this fact.

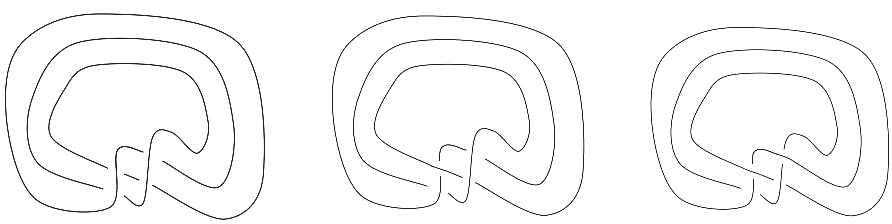

Figure 15 Crossing changes are applied to the diagram from Figure 14 (left) to produce the unknot (center) and a distinct, nontrivial knot (right). The original diagram on the left has $(c_1, c_2, c_3, c_4) = (-1, -1, -1, -1)$, the diagram in the center has $(c_1, c_2, c_3, c_4) = (+1, -1, -1, -1)$, and the diagram on the right has $(c_1, c_2, c_3, c_4) = (+1, -1, +1, -1)$, given the crossing labeling in Figure 14.

Suppose that the knotter plays first and suppose that the game has been played up until the last two moves. Since the unknotter has played twice by this point, she can ensure that regions 5 and 6 have already been played. The knotter has the second to last turn, for which there are two cases that need to be considered. Either the knotter makes a move that results in a nontrivial knot or not. If the knotter is unable to move in such a way that would produce a nontrivial knot, then the unknotter can keep the last region the same and win the game. Consider the second case where the knotter produces a nontrivial diagram in the second to last move. That is, the knotter creates a diagram where all crossings have the same sign or are alternating. If the unknotter performs an RCC move on region 1, 2, 3, or 4, she changes three of the four signs, since each of these regions is bounded by three crossings. Changing three signs in a homogeneous or alternating sign sequence results in a sign sequence that is neither homogeneous nor alternating. Thus, the unknotter wins playing second.

Now assume that the unknotter plays first, so our task is to demonstrate a winning strategy for the knotter. The unknotter can either play in the $\{1, 2\}$ pair, the $\{3, 4\}$ pair, or the $\{5, 6\}$ pair. Note that when we begin play, all crossings have the same sign. Hence, the crossing sequence is homogeneous and the diagram is knotted.

- Suppose that the unknotter plays in region 5 (resp. 6). The knotter should counteract this move by mirroring the unknotter's play in the corresponding region. If both regions 5 and 6 are kept the same, then the crossings in the diagram are unchanged. If both regions 5 and 6 are changed, then the diagram is the mirror image of the original. Thus, if the original diagram was knotted, it remains knotted.
- Suppose that the unknotter plays in region 1 (resp. 2). The knotter should counteract this move by mirroring the unknotter's play in region 2 (resp. 1). If both regions 1 and 2 are kept the same, then the signs of the crossings are maintained. If both regions 1 and 2 are changed, only crossings c_2 and c_4 change. So if the crossing sequence was homogeneous, it becomes alternating, and vice versa. In either case, the knottedness of the diagram is preserved.
- Suppose that the unknotter plays in region 3 (resp. 4). The knotter should counteract this move by mirroring the unknotter's play in region 4 (resp. 3). If both regions 3 and 4 are kept the same, then the signs of the crossings are maintained. If both regions 3 and 4 are changed, only crossings c_1 and c_3 change. So if the crossing sequence was homogeneous, it becomes alternating, and vice versa. Again, in either case, the knottedness of the diagram is preserved.

Since, for any move that can be made by the unknotter, the knotter has a response move that returns the knot diagram to a knotted state, the knotter always wins playing second. ■

In our investigation of torus knots, we were also able to prove the following results.

Proposition 4. *Suppose the Region Unknotting Game is played on a standard* $(2, 5)$-*torus knot diagram (as in Figure 13). Then the unknotter has the winning strategy both when playing first and when playing second.*

Proposition 5. *Suppose the Region Unknotting Game is played on a standard* $(2, 7)$-*torus knot diagram. Then the unknotter has the winning strategy when playing first.*

Our proofs of these propositions are long and arduous, so we omit them here. Instead, let's look at the big picture. The fact that the proofs of Propositions 4 and 5 are so difficult when approached head on* indicates that there is a lot more room for investigation and creativity when it comes to finding winning strategies in the Region Unknotting Game. With that in mind, we turn to our final thoughts.

*This is evidenced, in part, by the fact that we have not yet been able to find who has a winning strategy when the knotter plays first on the $(2, 7)$-torus knot.

What's next?

So far, we have collected the following open questions:

1. Which player has a winning strategy for the Region Unknotting Game when playing on a standard twist knot diagram with an odd number of crossings if the knotter plays first?
2. Which player has a winning strategy for the Region Unknotting Game when playing on a standard $(2, q)$-torus knot diagram, for q an odd integer greater than 5?

We feel that the answers to these two questions are related, since a $(2, q)$-torus knot diagram behaves somewhat like a twist knot diagram in the case where the knotter can ensure that the clasp remains clasped. Indeed, the crossings in a $(2, q)$-torus knot behave like the twist crossings in an odd twist knot diagram.†

In general, more progress needs to be made on various diagrams of torus knots. We should also expand our exploration to include standard diagrams of rational knots, a family of knots that includes twist knots and $(2, q)$-torus knots as some of its simplest members. Pretzel knots are another family of algebraic knots that would be fascinating to explore.

The investigation of the Region Unknotting Game has only just begun. We invite the reader to join us in playing more games, gaining deeper intuition, and inventing creative new ways of thinking about who should win the game, when they should be able to win, and how.

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Summary. Motivated by rcent work by A. Shimizu on a newly discovered unknotting operation and inspired by previous work on knot games, we introduce the Region Unknotting Game. We play the game on several types of knot diagrams, developing both our spatial intuition and our understanding of the structure of knots.

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† Notice that we only consider $(2, q)$-torus knots for q odd since the q even case corresponds to a link, not a knot.

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PINEMI PUZZLE

	7				5				2
	9	10		10	9		9	7	
7		8	6			9		11	
7			8	8		7	8		
			8		8		10		10
6		7							7
	7		8			10	8	11	
6				9		10		10	
	9		8		8				4
4				5				5	

How to play. Place one jamb (|), two jambs (||), or three jambs (|||) in each empty cell. The numbers indicate how many jambs there are in the surrounding cells—including diagonally adjacent cells. Each row and each column has 10 jambs. Note that no jambs can be placed in any cell that contains a number.

The solution is on page 394.

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Loci of Points Inspired by Viviani's Theorem

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Viviani (1622–1703), who was a student and assistant of Galileo, discovered that equilateral triangles satisfy the following property: The sum of the distances from the sides of any point inside an equilateral triangle is constant.

Viviani's theorem can be easily proved by using areas. Joining a point P inside the triangle to its vertices divides it into three parts. The sum of their areas will be equal to the area of the original one. Therefore, the sum of the distances from the sides will be equal to the altitude, and the theorem follows.

Given a triangle (which includes both boundary and inner points), Δ, and a positive constant, k, we shall consider the following questions:

Question 1. *What is the locus $T_k(\Delta)$ of points P in Δ such that the sum of the distances of P to each of the sides of Δ equals k?*

Question 2. *What is the locus $S_k(\Delta)$ of points P in the plane such that the sum of the squares of the distances of P to each of the sides of Δ equals k?*

By Viviani's theorem, the answer to the first question is straightforward for equilateral triangles: If k equals the altitude of an equilateral triangle, then the locus of points is the entire triangle. On the other hand, if k is not equal to the altitude of the equilateral triangle, then the locus of points is empty.

Since the distance function is linear with two variables and because of squaring the distances, the locus of points in the second question, if not empty, is a quadratic curve. In our case, we shall see that this quadratic curve is always an ellipse. This fact will be the basis of a new characterization of ellipses.

We start first with the locus of points that have constant sum of distances to each of the sides of a given triangle and consider separately the cases of isosceles and scalene triangles. Then, we turn to the locus of points that have constant sum of squared distances to each of the sides of a given triangle. We prove the main theorem that gives a characterization of ellipses and conclude by discussing the minimal sum of squared distances.

Constant sum of distances

Samelson [**6**, p. 225] gave a proof of Viviani's theorem that uses vectors and Chen & Liang [**3**, p. 390–391] used this vector method to prove a converse: If inside a triangle there is a circular region in which the sum of the distances from the sides is constant, then the triangle is equilateral. In [**2**], this converse is generalized in the form of two theorems that we will use in addressing Question 1. To state these theorems, we need the following terminology: Let $\mathcal{P}$ be a polygon consisting of both boundary and interior points. Define the *distance sum function* $\mathcal{V} : \mathcal{P} \to \mathbb{R}$, where for each point $P \in \mathcal{P}$ the value $\mathcal{V}(P)$ is defined as the sum of the distances from P to the sides of $\mathcal{P}$.

Theorem 1. *Any triangle can be divided into parallel line segments on which $\mathcal{V}$ is constant. Furthermore, the following conditions are equivalent:*

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MSC: Primary 51N20

- *$\mathcal{V}$ is constant on the triangle Δ.*
- *There are three noncollinear points, inside the triangle, at which $\mathcal{V}$ takes the same value.*
- *Δ is equilateral.*

Theorem 2. *(a) Any convex polygon $\mathcal{P}$ can be divided into parallel line segments, on which $\mathcal{V}$ is constant.*
(b) If $\mathcal{V}$ takes equal values at three noncollinear points, inside a convex polygon, then $\mathcal{V}$ is constant on $\mathcal{P}$.

The discussion in [**2**, p. 207–210] lays out a connection between linear programming and Theorems 1 and 2. Theorem 1 is proved explicitly by defining a suitable linear programming problem, and Theorem 2 is proved by means of analytic geometry, where the distance sum function $\mathcal{V}$ is computed directly and shown to be linear in two variables. We shall apply these theorems to find the loci $T_k(\Delta)$ for isosceles and scalene triangles Δ.

Isosceles triangle For isosceles triangles, we exploit their reflection symmetry to find directly the line segments on which $\mathcal{V}$ is constant, and for scalene triangles, we apply either of the above theorems to show that $\mathcal{V}$ is constant on certain parallel line segments. To find the direction of these line segments, we compute explicitly the equation of $\mathcal{V}$. The idea is illustrated through an example.

Since an isosceles triangle has reflection symmetry across the altitude, we conclude that: If the sum of distances of point P from the sides of the triangle is k, then the reflection point P' across the altitude satisfies the same property. For such triangles, we have the following proposition.

Proposition 1. *If Δ is a nonequilateral isosceles triangle and k ranges between the lengths of the smallest and the largest altitudes of the triangle, then $T_k(\Delta)$ is a line segment, whose end points are on the boundary of the triangle, parallel to the base.*

Proof. In Figure 1, if the line segment DE is parallel to the base BC, then, when P moves along DE, the length a remains constant and $b + c = DP \sin\alpha + PE \sin\alpha = DE \sin\alpha$, which is also constant. Hence, $a + b + c = k$ is constant on the segment DE. By Theorem 1, there are no other points in Δ with distance sum k unless the triangle is equilateral.

If a approaches zero, then the length of DE approaches the length of the base BC, and $k = a + DE \sin\alpha$ approaches the length of the altitudes from B (or C). On the other hand, if the length of DE approaches zero, then k approaches the length of the altitude from A. Hence, $T_k(\Delta)$ is defined whenever k ranges between the lengths of the smallest and the largest altitudes of the triangle. ■

Scalene triangle Similarly, by Theorem 1, the locus of points $T_k(\Delta)$ for a scalene triangle Δ is a line segment, provided k ranges between the lengths of the smallest and the largest altitudes of the triangle. Otherwise, $T_k(\Delta)$ will be empty. Indeed, the connection with linear programming allows us to deduce the result. The convex polygon is taken to be the feasible region, and the distance sum function $\mathcal{V}$ corresponds to the objective function. The parallel line segments, on which $\mathcal{V}$ is constant, correspond to isoprofit lines. The mathematical theory behind linear programming states that an optimal solution to any problem will lie at a corner point of the feasible region. If the feasible region is bounded, then both the maximum and the minimum are attained at corner points. Now, the distance sum function $\mathcal{V}$ is a linear continuous function in two variables. The values of $\mathcal{V}$ at the vertices of the feasible region, which is the triangle

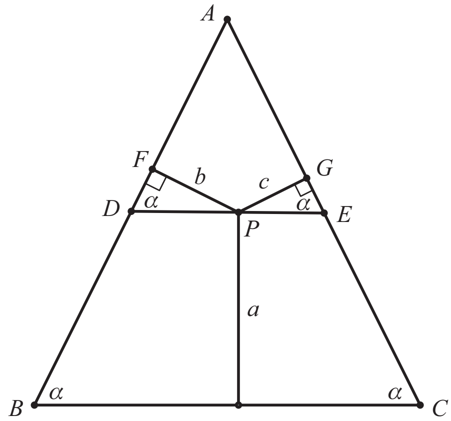

Figure 1 The locus of points is a line segment parallel to the base.

Δ, are exactly the lengths of the altitudes. Moreover, this function attains its minimum and its maximum at the vertices of the triangle and ranges continuously between its extremal values. Hence, **it takes on every value between its minimum and its maximum**. Therefore, we have the following proposition.

Proposition 2. *If Δ is a scalene triangle and k ranges between the lengths of the smallest and the largest altitudes of the triangle, then $T_k(\Delta)$ is a line segment, whose end points are on the boundary of the triangle.*

The question is: How can we determine this segment for a general triangle? We shall illustrate the method by the following example. Given the vertices of a triangle, we first compute the equations of the sides then we find the distances from a general point (x, y) inside the triangle to each of the sides. In this way, we obtain the corresponding distance sum function $\mathcal{V}$. Taking $\mathcal{V} = c$, we get a family of parallel lines that, for certain values of the constant c, intersect the given triangle in the desired line segments.

Example 1. Let Δ be the right-angled triangle with vertices (0, 0), (0, 3), and (4, 0), respectively (see Fig. 2). If the constant sum of distances from the sides is k, $2.4 \leq k \leq 4$, then the locus is a line segment inside the triangle parallel to the line $2x + y = 0$.

In Fig. 2, $T_{2.4}(\Delta) = \{A\}$, $T_{2.8}(\Delta) = DG$, $T_{3.2}(\Delta) = EH$, $T_{3.6}(\Delta) = FI$ and $T_4(\Delta) = \{C\}$. Note that at the extremal values of k, the segments shrink to a corner point of the triangle Δ.

Indeed, the smallest altitude of the triangle is 2.4, and the largest altitude is 4. Hence, by the previous proposition, the locus of points is a line segment. To find this line segment, we compute the distance sum function $\mathcal{V}$. The equation of the hypotenuse is $3x + 4y = 12$. Hence,

$$\mathcal{V} = x + y - \frac{3x + 4y - 12}{5} = \frac{2}{5}x + \frac{1}{5}y + \frac{12}{5}.$$

Therefore, the lines $\mathcal{V} = c$ are parallel to the line $2x + y = 0$, and the result follows.

Note that the proofs of Theorems 1 and 2 follow the same line.

Constant sum of squares of distances

Motivated by the previous results, we deal with the second question of finding the locus of points that have a constant sum of squares of distances from the sides of a

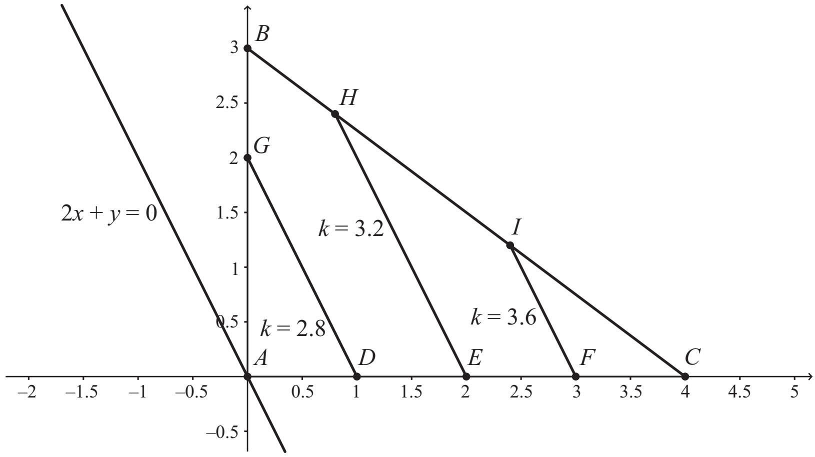

Figure 2 $T_k(\Delta)$, $2.4 \le k \le 4$, are line segments inside the triangle parallel to the line $2x + y = 0$.

given triangle. Referring to Example 1, $DE^2 + DF^2 + DG^2 = 5$ (the number 5 is chosen arbitrary) implies the following equation of a quadratic curve;

$$x^2 + y^2 + \left(\frac{3x + 4y - 12}{5}\right)^2 = 5.$$

Simplifying, one gets the equivalent equation

$$34x^2 + 41y^2 + 24xy - 72x - 96y + 19 = 0.$$

This is exactly the equation of the ellipse shown in Fig. 3.

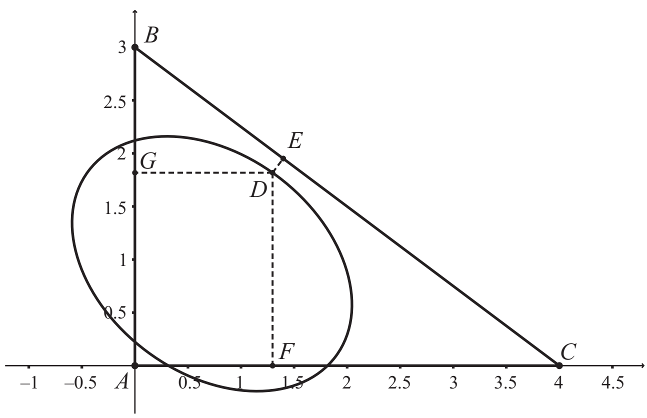

Figure 3 Each point D on the ellipse satisfies $DE^2 + DF^2 + DG^2 = 5$.

A new characterization of the ellipse In view of the previous example, we may prove the following theorem, which states that among all quadratic curves the ellipse can be characterized as the locus of points that have a constant sum of squares of distances from the sides of an appropriate triangle.

Theorem 3. *The locus $S_k(\Delta ABC)$ of points that have a constant sum of squares of distances from the sides of an appropriate triangle ABC is an ellipse (and vice versa).*

Proof. We shall prove the following two claims:

(a) Given a triangle, the locus of points that have a constant sum of squares of distances from the sides is an ellipse. These loci, for different values of the constant, are homothetic ellipses with respect to their common center (their corresponding axes are proportional with the same factor).

(b) Given an ellipse, there is a triangle for which the sum of the squares of the distances from the sides, for all points on the ellipse, is constant.

Using analytic geometry, we choose for part (a) the coordinate system such that the vertices of the triangle lie on the axes. Suppose the coordinates of the vertices of the triangle are $A(0,a)$, $B(-b,0)$, $C(c,0)$, where $a,b,c>0$ (see Fig. 4). Let (x,y) be any point in the plane, and let d_1, d_2, d_3 be the distances of (x,y) from the sides of the triangle ABC. It follows that

$$\sum_{i=1}^{3} d_i^2 = \frac{(ax+cy-ac)^2}{a^2+c^2} + \frac{(ax-by+ab)^2}{a^2+b^2} + y^2.$$

Hence, $\sum_{i=1}^{3} d_i^2 = k$ (constant) if and only if the point (x,y) lies on the quadratic curve

$$\frac{(ax+cy-ac)^2}{a^2+c^2} + \frac{(ax-by+ab)^2}{a^2+b^2} + y^2 = k. \tag{1}$$

In general, a quadratic equation in two variables,

$$\mathcal{A}x^2 + \mathcal{B}xy + \mathcal{C}y^2 + \mathcal{D}x + \mathcal{E}y + \mathcal{F} = 0, \tag{2}$$

represents an ellipse provided the discriminant $\delta = \mathcal{B}^2 - 4\mathcal{A}\mathcal{C} < 0$ is negative.

To prove our claim, note that from equation (1) we have

$$\mathcal{A} = \frac{a^2}{p} + \frac{a^2}{q}, \mathcal{B} = \frac{2ac}{p} - \frac{2ab}{q}, \mathcal{C} = \frac{c^2}{p} + \frac{b^2}{q} + 1 \tag{3}$$

where $p = a^2 + c^2$ and $q = a^2 + b^2$. Therefore,

$$\delta = \mathcal{B}^2 - 4\mathcal{A}\mathcal{C} = -4\frac{a^2}{pq}\left(b^2 + 2bc + c^2 + p + q\right),$$

and the result follows.

Now, we prove part (b). Applying rotation or translation of the axes, we may assume that the ellipse has a canonical form $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$, $\alpha \ge \beta > 0$. If we find positive real numbers a, b with $\alpha^2 = a^2 + 3b^2$, $\beta^2 = 2a^2$, then the isosceles triangle with vertices $A'(0, a-l)$, $B'(-b,-l)$, $C'(b,-l)$, gives the required property in part (b), where $l = \frac{2ab^2}{a^2+3b^2}$.

Indeed, because of the symmetry of the ellipse, we choose first an isosceles triangle with vertices $A(0,a)$, $B(-b,0)$, $C(b,0)$ and compute the sum of the squared distances from its sides. Substituting $c=b$ in equation (1) gives the equation of the locus $S_k(\Delta ABC)$ as

$$\frac{(ax+by-ab)^2}{a^2+b^2} + \frac{(ax-by+ab)^2}{a^2+b^2} + y^2 = k.$$

Equivalently, we get a translation of a canonical ellipse:

$$\frac{x^2}{a^2+3b^2} + \frac{(y-\frac{2ab^2}{a^2+3b^2})^2}{2a^2} = \frac{(a^2+b^2)k - 2a^2b^2}{2a^2(a^2+3b^2)} + \frac{2b^4}{(a^2+3b^2)^2}. \tag{4}$$

TABLE 1: Examples

a	b	$\alpha^2 = a^2 + 3b^2$	$\beta^2 = 2a^2$	Ellipse	k	Figure
1	1	4	2	$\frac{x^2}{4} + \frac{y^2}{2} = 1$	$\frac{9}{2}$	5
$\sqrt{3}$	1	6	6	$x^2 + y^2 = 6$	10	6

Thus, it is enough to take

$$\frac{(a^2+b^2)k - 2a^2b^2}{2a^2(a^2+3b^2)} + \frac{2b^4}{(a^2+3b^2)^2} = 1,$$

and therefore,

$$k = \frac{2a^2(a^4 + 7a^2b^2 + 10b^4)}{(a^2+b^2)(a^2+3b^2)}. \tag{5}$$

Translating downward by $l = \frac{2ab^2}{a^2+3b^2}$, we get that the ellipse $\frac{x^2}{a^2+3b^2} + \frac{y^2}{2a^2} = 1$ is the locus of points, $S_k(\Delta A'B'C')$, which have a constant sum of squares of distances from the sides of the triangle with vertices $A'(0, a-l)$, $B'(-b, -l)$, $C'(b, -l)$. Moreover, this constant is given by equation (5). ∎

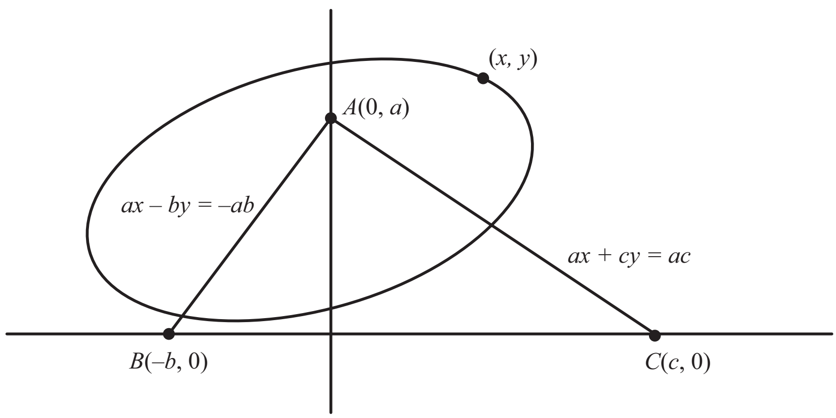

Figure 4 The locus is an ellipse.

Table 1 includes two examples that demonstrate Theorem 3.

Notice that, though a given triangle and a given constant k yield at most one ellipse, the same ellipse can be obtained using different triangles. This can be easily seen in Fig. 5. By the above computations, the ellipse $\frac{x^2}{4} + \frac{y^2}{2} = 1$ is the locus of points $S_{4.5}(\Delta)$, where Δ is the triangle with vertices $A' = (0, 0.5)$, $B' = (-1, -0.5)$ and $C' = (1, -0.5)$. If we reflect the whole figure across the $x-$axis, then the same ellipse $\frac{x^2}{4} + \frac{y^2}{2} = 1$ will be the locus of points $S_{4.5}(\widetilde{\Delta})$, where $\widetilde{\Delta}$ is the reflection of Δ across the $x-$axis with vertices: $(0, -0.5)$, $(-1, 0.5)$, and $(1, 0.5)$.

Notice also that, in the second example, the locus of points is a circle. In general, the locus of points is a circle exactly when, in equation (2),

$$\mathcal{A} = \mathcal{C} \text{ and } \mathcal{B} = 0.$$

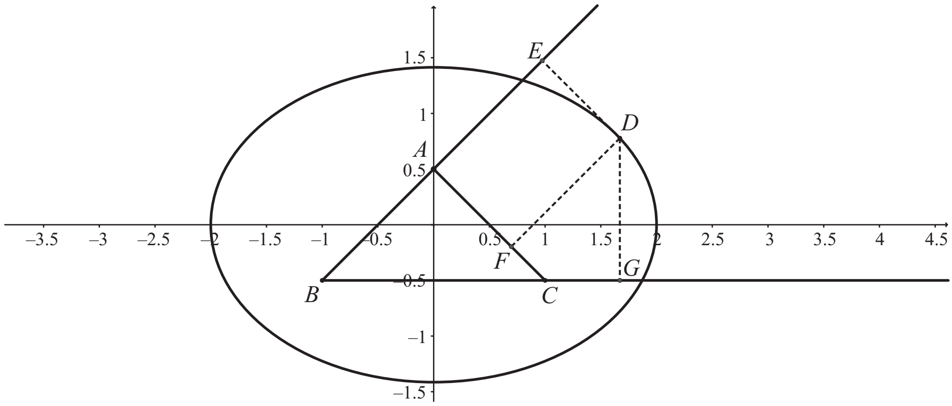

Figure 5 The locus is an ellipse with $k = 4.5$.

Equivalently, from (3) we have

$$\frac{a^2}{p} + \frac{a^2}{q} = \frac{c^2}{p} + \frac{b^2}{q} + 1 \quad \text{and} \quad \frac{2ac}{p} - \frac{2ab}{q} = 0. \tag{6}$$

Substituting the values of p and q and simplifying, we get

$$\frac{2ac}{p} - \frac{2ab}{q} = 0 \Leftrightarrow (a^2 - bc)(c - b) = 0.$$

Consequently, two cases have to be considered:

First, $a^2 = bc$, in which case, the triangle ΔABC is right angled. This case does not occur since the conditions in (6) lead to a contradiction as follows: $\frac{2ac}{p} - \frac{2ab}{q} = 0$ implies $\frac{c}{p} = \frac{b}{q}$ or equivalently $\frac{bc}{p} = \frac{b^2}{q}$. Since $a^2 = bc$, we get $\frac{a^2}{p} = \frac{b^2}{q}$. Hence, the first condition in (6) simplifies into $\frac{a^2}{q} = \frac{c^2}{p} + 1$. This equation together with the relations $a^2 = bc$ and $\frac{c}{p} = \frac{b}{q}$ yield a contradiction.

Second, $c = b$, in which case the triangle ΔABC is isosceles. In this case, we have $p = q$. Substituting in the first condition of (6) we get $a^2 + 3b^2 = 2a^2$, which is equivalent to $a = \sqrt{3}b$. Hence, the vertices of the triangle are $A(0, \sqrt{3}b)$, $B(-b, 0)$ and $C(b, 0)$. This implies that the triangle is equilateral.

Therefore, we have the following result.

Conclusion 1. *The locus $S_k(\Delta)$ of points that have a constant sum of squares of distances from the sides of a given triangle Δ is a circle if and only if the triangle Δ is equilateral.*

Minimal sum of squared distances Another interesting question is to find the minimal sum of squared distances from the sides of a given triangle. For the isosceles triangle with vertices $A(0, a)$, $B(-b, 0)$, and $C(b, 0)$, equation (4) indicates that, as the right-hand side approaches 0, the ellipse degenerates to one point: $(0, \frac{2ab^2}{a^2+3b^2})$. Thus, the locus of points is defined exactly when

$$\frac{(a^2 + b^2)k - 2a^2b^2}{2a^2(a^2 + 3b^2)} + \frac{2b^4}{(a^2 + 3b^2)^2} \geq 0.$$

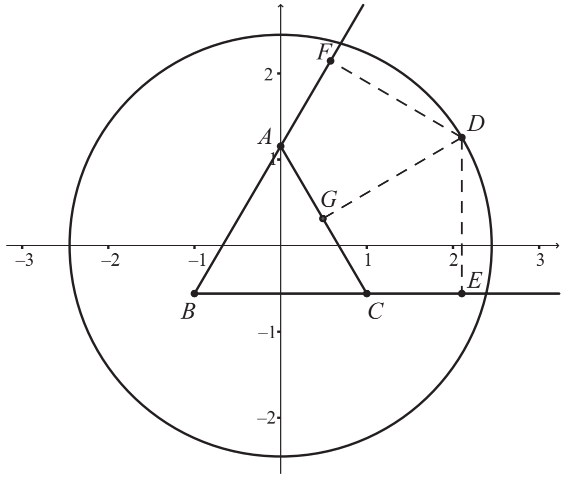

Figure 6 The locus is a circle with $k = 10$.

Equivalently,

$$k \geq \frac{2a^2b^2}{a^2 + 3b^2}.$$

Therefore, the following consequence holds.

Conclusion 2. *For the isosceles triangle with vertices $A(0, a)$, $B(-b, 0)$, and $C(b, 0)$, the minimal sum of squared distances from the sides is $\frac{2a^2b^2}{a^2+3b^2}$ attained at the point $(0, \frac{2ab^2}{a^2+3b^2})$ inside the triangle.*

When $a^2 = 3b^2$, then the triangle is equilateral with vertices $A(0, \sqrt{3}b)$, $B(-b, 0)$, and $C(b, 0)$. In this case, the minimal sum of squared distances is b^2 attained at the point $(0, \frac{\sqrt{3}}{3}b)$, which is exactly the incenter of the equilateral triangle. In Fig. 7, $b = 1$

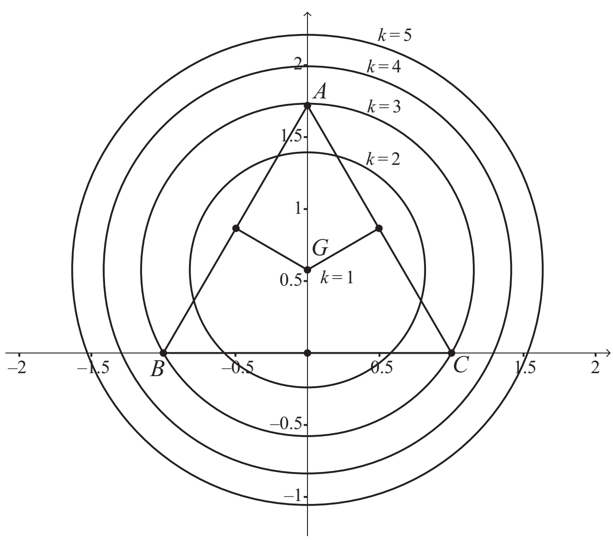

Figure 7 The circles degenerate to the incenter G of the equilateral triangle, where the minimum of k is 1.

and the loci of points are circles that degenerate to the incenter of the equilateral triangle as k approaches 1.

Concluding remarks

Other related results are the following: Kawasaki [**4**, p. 213], with a proof without words, used only rotations to establish Viviani's theorem. Polster [**5**], considered a natural twist to a beautiful one-glance proof of Viviani's theorem and its implications for general triangles.

The restriction of the locus, in the first question, to subsets of the closed triangle is intended to avoid the use of the signed distances. These signed distances, when considered, allow us to search loci of points outside the triangle for "large" values of the constant k. The reader is encouraged to do some examples.

Theorem 2, allows us to address Question 1 to any convex polygon in the plane.

The question about loci of points, with constant sum of squares of distances, can be generalized to any polygon or any set of lines in the plane. In this case, one should distinguish whether all the lines are parallel or not. The reader is encouraged to work out some examples using GeoGebra.

Conclusion 2 can be restated for general triangles by a change of the coordinate system. This demands familiarity with the following theorem (which is beyond our discussion): Every real quadratic form $q = X^T AX$ with symmetric matrix A can be reduced by an orthogonal transformation to a canonical form.

Finally, this characterization of the ellipse was exploited to build an algorithm for drawing ellipses using GeoGebra (see [**1**]).

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Summary. We consider loci of points such that their sum of distances or sum of squared distances to each of the sides of a given triangle is constant. These loci are inspired by Viviani's theorem and its extension. The former locus is a line segment or the whole triangle and the latter locus is an ellipse.

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The Osculating Circle Without the Unit Normal Vector

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One important technique in modern mathematics is to approximate complicated mathematical objects by simpler ones. Perhaps the best known instance of such approximations in calculus is provided by tangent lines: When we find the tangent lines of a plane curve, we approximate it in various points by straight lines, which are the simplest curves we know in the plane.

Although the tangent line of a plane curve has the same derivative as the curve itself at the point they meet one another, there is one problem with the tangent line approximation that motivates us to think of a better substitution. This is the fact that the tangent line and the curve may fail to have the same curvature at their common point. More precisely, while the curvature of any straight line is 0, the curve we approximate may have nonzero curvature at many points.

To overcome this deficiency, we may use osculating circles instead of tangent lines to obtain better approximations of plane curves. When P is a point on some curve C at which C has nonzero curvature κ, the *osculating circle* (OC) of C at P is a circle that passes through P, whose radius is $1/\kappa$ and whose center is located in the inner side of C and on the normal line at P (see Figure 1). Note that in some books, like [**2**, pp. 874], osculating circles are referred to as *circle of curvature*. Gottfried Wilhelm Leibniz named them *circulus osculans*, which is the Latin for *kissing circle*.

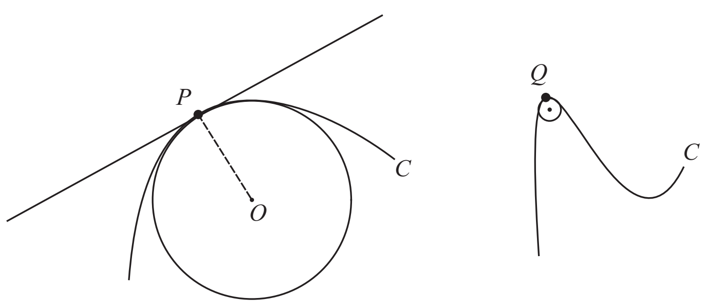

Figure 1 Instances of osculating circles.

The main idea behind osculating circles is to inscribe circles inside plane curves to find approximations which are better than the linear ones. Of course, as we mentioned above, this *"circular approximation"* can be only used at those points at which the curve has nonzero curvature, for otherwise a linear approximation would be the only possible one.

In Figure 1, two curves are drawn together with one osculating circle for each. As can be seen in the figure, the osculating circle at the point P of the curve C provides a better approximation of the curve than that provided by the tangent line at the same

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point. This is because, by the definition, the circle has the same curvature (besides having the same tangent line) as the curve C at P. To understand why, recall that the curvature of a circle is the reciprocal of its radius, so when we defined the radius of the OC to be $1/\kappa$, the curvature of the circle would be κ.

Note that by our definition, the radius of the OC at a point where the curve has great curvature must be small, and vice versa. This is also illustrated in Figure 1. In fact, while the curvature of C' at Q is more than that of C at P, the radius of the OC at Q is less than that in P.

The OC is usually introduced within the context of vector functions. In fact, to find the OC of a parametric curve

$$x = x(t),\ y = y(t); \quad a \leq t \leq b$$

at some point $P_0 = (x(t_0), y(t_0))$, $a < t_0 < b$, one considers the curve as the graph of a vector function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}; \quad a \leq t \leq b,$$

and then uses vector techniques to find both the radius and center of the OC. The details of this approach to the OC can be found in any standard calculus text. See [**1**, pp. 645], [**4**, pp. 939], and [**3**, pp. 883] for example.

The curvature of the above-mentioned parametric curve at $(x(t_0), y(t_0))$, obtained by vector methods, is given by

$$\kappa(t_0) = \frac{|x'(t_0)y''(t_0) - x''(t_0)y'(t_0)|}{\left((x'(t_0))^2 + (y'(t_0))^2\right)^{3/2}}.$$

Here it is assumed that the functions x and y are twice differentiable at t_0, and that $x'(t_0)$ and $y'(t_0)$ are not both equal to zero. Now, having our above discussion in mind, we find that the radius of the OC at $(x(t_0), y(t_0))$ is

$$R(t_0) = \frac{1}{\kappa(t_0)} = \frac{\left((x'(t_0))^2 + (y'(t_0))^2\right)^{3/2}}{|x'(t_0)y''(t_0) - x''(t_0)y'(t_0)|}, \tag{1}$$

provided that $\kappa(t_0) \neq 0$.

Another important ingredient of the OC is its center. This is, to the best of our knowledge, always found using *unit normal vectors*. This is because the center of the OC is defined to be located on the inner normal line of the curve, and the unit normal vector at each point is directed towards the inner side of the curve (see [**4**, Section 13.4] for the definition of the unit normal vector). Figure 2 illustrates the way we can use the unit normal vector to find the center of the OC. We just need to multiply the vector by the radius of the OC, and then add the resulting vector to the one that represents the point under consideration.

In the context of Figure 2, the point under study is (0, 0). By (1), the radius of the OC at this point is $\frac{1}{2}$. Also, the unit normal vector at this point is $\mathbf{j}$. Now, the center of the OC at (0, 0), namely $(0, \frac{1}{2})$, is the endpoint of the vector $\mathbf{0} + \frac{1}{2}\mathbf{j}$.

As another example, consider the circle

$$x = r\cos t,\ y = r\sin t; \quad 0 \leq t \leq 2\pi,\ r > 0.$$

It follows from (1) that the radius of the OC at every point is r, that is, the radius of the circle itself. Therefore, the OC at each point of a circle is the circle itself. Notice the analogy of this to the fact that the tangent line at every point of a straight line is the line itself.

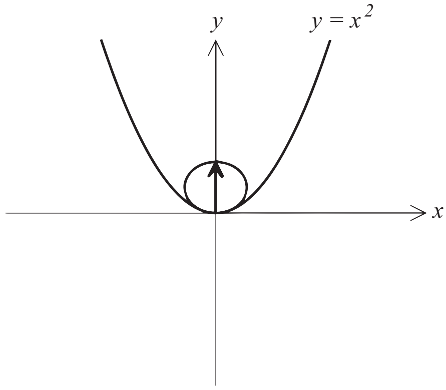

Figure 2 The graph of $y = x^2$ with its osculating circle and unit normal vector at (0, 0).

Finding the center of the osculating circle

The vector approach to the center of the OC is sometimes confusing for students, and often requires considerable calculations. Our aim in the sequel is to show that once the radius of an OC is determined, its center can also be found using the theory of parametric plane curves. To this end, let us fix a plane curve given by the parametric equations

$$x = x(t), \;\; y = y(t); \quad a \le t \le b.$$

We want to find the center $O = (u, v)$ of the OC at a point $P_0 = (x(t_0), y(t_0))$, $a < t_0 < b$.

For the sake of simplicity, let us write x_0 instead of $x(t_0)$, x_0' for $x'(t_0)$, and so on. With this notation, (1) says that the radius of the OC at P_0 is

$$R_0 = \frac{\big((x_0')^2 + (y_0')^2\big)^{3/2}}{|x_0'y_0'' - x_0''y_0'|}. \tag{2}$$

Also, we recall that the curve we considered is concave upward at P_0 if the number

$$\frac{x_0'y_0'' - x_0''y_0'}{(x_0')^3} \tag{3}$$

is greater than zero, and it is concave downward there when this number is less than zero, provided that $x_0' \neq 0$. This formula follows from the last formula on page 669 of [**3**], where the second derivative of a curve defined parametrically is given.

Before proceeding, we should emphasize that to make the OC well-defined, both the numerator and denominator of the fraction in (2) must be nonzero. For this reason, we will assume in what follows that

(i) the numbers x_0' and y_0' satisfy $(x_0')^2 + (y_0')^2 \neq 0$;
(ii) the number $x_0'y_0'' - x_0''y_0'$ is not zero.

Now, to find the point O, we consider two cases as follows.

1. $x_0' = 0$. In this case, the curve has a horizontal normal line at P_0. So, v is equal to y_0, while u can be either greater or smaller than x_0. In fact, depending on the direction of the inner side of the curve, u is either $x_0 + R_0$ or $x_0 - R_0$. This is illustrated in Figure 3.

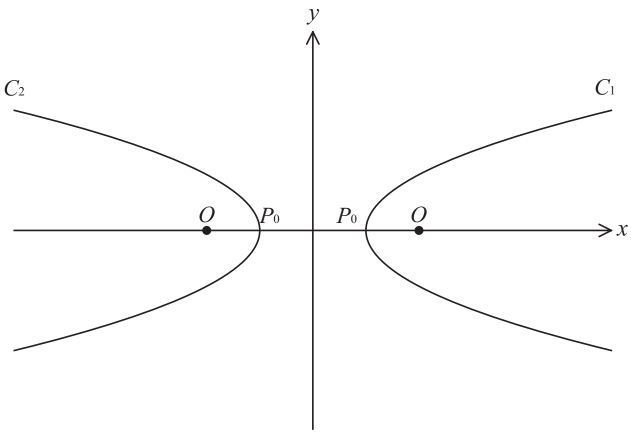

Figure 3 For C_1, $O = (x_0 + R_0, y_0)$, and for C_2, $O = (x_0 - R_0, y_0)$.

Notice that by our assumption (*ii*) above, $x_0' = 0$ implies that $x_0'' \neq 0$. We claim that when $x_0'' > 0$, $O = (x_0 + R_0, y_0)$, and when $x_0'' < 0$, $O = (x_0 - R_0, y_0)$. To see this, note that a 90° counterclockwise rotation of the curve gives us

$$x_1(t) = -y(t),\ \ y_1(t) = x(t); \quad a \le t \le b.$$

For this new curve, (3) simplifies to $x_0''/(y_0')^2$. This shows that when $x_0'' > 0$, the new curve is concave upward at

$$(x_1(t_0), y_1(t_0)) = (-y_0, x_0),$$

so that $(-v, u)$, the point obtained from O by the rotation, lies above $(-y_0, x_0)$. This shows that in this case $u > x_0$, and we must have $O = (x_0 + R_0, y_0)$. Since putting $x_0' = 0$ in (2) yields

$$R_0 = \frac{(y_0')^2}{|x_0''|},$$

we find that

$$O = \left(x_0 + \frac{(y_0')^2}{x_0''}, y_0\right). \tag{4}$$

A similar discussion applies if $x_0'' < 0$, indicating that O is also given by (4) in this case.

2. $x_0' \neq 0$. In this case, the coordinates u and v of O satisfy

$$x_0'(u - x_0) + y_0'(v - y_0) = 0 \tag{5}$$

and

$$\sqrt{(u - x_0)^2 + (v - y_0)^2} = \frac{\left((x_0')^2 + (y_0')^2\right)^{3/2}}{|x_0' y_0'' - x_0'' y_0'|} \tag{6}$$

as O lies on the normal line at P_0, and the distance between O and P_0 is R_0. It follows from (5) that

$$(u - x_0)^2 = \left(\frac{y_0'}{x_0'}\right)^2 (v - y_0)^2.$$

Putting the right-hand side of this for $(u - x_0)^2$ in (6) and then squaring both sides we find, by a simple calculation, that v is either

$$y_0 + x_0' \frac{(x_0')^2 + (y_0')^2}{x_0' y_0'' - x_0'' y_0'}$$

or

$$y_0 - x_0' \frac{(x_0')^2 + (y_0')^2}{x_0' y_0'' - x_0'' y_0'}.$$

Thus, from (5) we find that the points

$$P_1 = \left(x_0 + y_0' \frac{(x_0')^2 + (y_0')^2}{x_0' y_0'' - x_0'' y_0'}, y_0 - x_0' \frac{(x_0')^2 + (y_0')^2}{x_0' y_0'' - x_0'' y_0'} \right)$$

and

$$P_2 = \left(x_0 - y_0' \frac{(x_0')^2 + (y_0')^2}{x_0' y_0'' - x_0'' y_0'}, y_0 + x_0' \frac{(x_0')^2 + (y_0')^2}{x_0' y_0'' - x_0'' y_0'} \right)$$

are our only candidates for being the center of the OC at P_0.

If the curve is concave upward (resp., downward) at P_0, then the osculating circle at P_0 lies above (resp., below) the tangent line, therefore the y-component of O must be greater (resp., less) than y_0. This is illustrated in Figure 4.

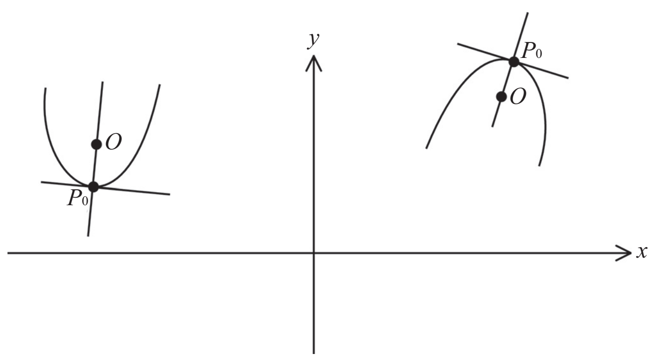

Figure 4 Comparing the y-component of O and P_0.

But, if

$$\frac{x_0' y_0'' - x_0'' y_0'}{x_0'} > 0,$$

then the curve is concave upward at P_0 and

$$y_0 + \frac{x_0'}{x_0' y_0'' - x_0'' y_0'} ((x_0')^2 + (y_0')^2) > y_0.$$

Similarly, when

$$\frac{x_0' y_0'' - x_0'' y_0'}{x_0'} < 0,$$

the curve is concave downward at P_0 and

$$y_0 + \frac{x_0'}{x_0'y_0'' - x_0''y_0'}((x_0')^2 + (y_0')^2) < y_0.$$

This shows that in both cases, $O = P_2$.

Finally, we note that (4) can be obtained from P_2 by letting $x_0' = 0$. This shows that the center of the osculating circle of the curve

$$x = x(t), \;\; y = y(t); \quad a \le t \le b$$

at P_0 is, in general, given by

$$O = \left(x_0 - y_0' \frac{(x_0')^2 + (y_0')^2}{x_0'y_0'' - x_0''y_0'}, \, y_0 + x_0' \frac{(x_0')^2 + (y_0')^2}{x_0'y_0'' - x_0''y_0'} \right).$$

Acknowledgment The author thanks the referees for their invaluable comments.

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Summary. The center of an osculating circle of a plane curve is usually found via the unit normal vector. In this paper we show that, once the radius of this circle is given, the center can be found using the calculus of parametric plane curves. This approach may be instructive for students.

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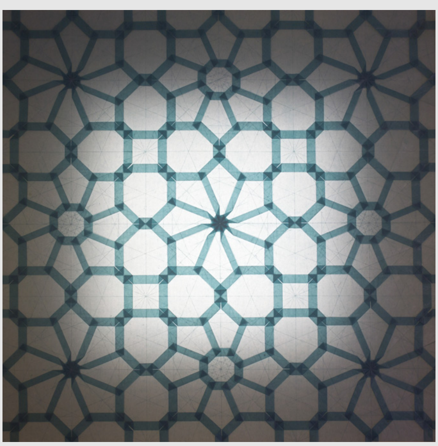

Artist Spotlight: Chris K. Palmer

Zillij Octagrams, © Chris K. Palmer; uncut square paper fold, c. 1994. A traditional tiling interpreted with pleats. Image courtesy of the artist.

See interview on page 380.

An Infinite Series that Displays the Concavity of the Natural Logarithm

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The inequality $\log x < x - 1$, which holds for all positive real $x \neq 1$ [**2**, Theorem 150, p. 106] is a fundamental but often overlooked property of the natural logarithm. In fact, it is equivalent to concavity. To see this, fix $a > 0$, replace x by x/a, and note that if $0 < x \neq a$, then $\log x - \log a = \log(x/a) < (x/a) - 1 = (1/a)(x - a)$.

On the other hand, the tangent line to the graph of the equation $y = \log x$ at the point $(a, \log a)$ has equation $y - \log a = (x - a)\log' a = (1/a)(x - a)$, so $y > \log x$ and the tangent line lies above the curve. Concavity in the form of the inequality $\log x < x - 1$ is known to imply the familiar inequality between the arithmetic and geometric means of a finite set of positive real numbers [**2**, p. 119].

The inequality $\log x < x - 1$ for $0 < x \neq 1$ is an immediate consequence of the integral representation for the natural logarithm, via the identity

$$x - 1 - \log x = \int_1^x \int_1^t u^{-2}\, du\, dt, \tag{1}$$

since the double integral (1) is obviously positive for all positive real $x \neq 1$. Although the Taylor series representation

$$x - 1 - \log x = \sum_{k=2}^{\infty} \frac{(1-x)^k}{k} \qquad (0 < x \le 2) \tag{2}$$

can be used to show that $\log x < x - 1$ for $0 < x < 1$ and $1 < x \le 2$, the series (2) diverges for $x > 2$. Therefore, it is interesting to note that there is a representation which, rather than employing integrals, or series which converge only on limited portions of the domain, expresses the difference $x - 1 - \log x$ as an infinite series which converges for all $x > 0$ and which has the property that each term is obviously positive for $0 < x \neq 1$. Equivalently, we prove the following result.

Proposition. *For every positive real x, the natural logarithm of x has the convergent infinite series representation*

$$\log x = x - 1 - \sum_{k=1}^{\infty} 2^{k-1}\left(x^{2^{-k}} - 1\right)^2. \tag{3}$$

Proof. First, observe that for every positive real x, the representation (3) can be written in the equivalent form

$$\log x = x - 1 - \sum_{k=1}^{\infty} \frac{1}{2^{k+1}}\left(\frac{x^{2^{-k}} - 1}{2^{-k}}\right)^2. \tag{4}$$

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MSC: Primary 33B10

Since for all real $x > 0$,

$$\lim_{k\to\infty}\left(\frac{x^{2^{-k}}-1}{2^{-k}}\right)^2 = (\log x)^2 < \infty,$$

the infinite series in (4) (and hence also in (3)) converges by limit comparison with the convergent geometric series $\sum_{k=1}^{\infty}\frac{1}{2^{k+1}}$.

To complete the proof, fix $x > 0$ and employ the definition of the derivative

$$\log x = \frac{d}{dh}x^h\bigg|_{h=0} = \lim_{h\to 0}\frac{x^h-1}{h} = \lim_{h\to 0} h^{-1}\big(x^h-1\big) = \lim_{n\to\infty} 2^n\big(x^{2^{-n}}-1\big), \tag{5}$$

as a limit of a difference quotient [**1**, Ch. III, §6, pp. 175–176]. Using (5), we can express $x-1-\log x$ for $x > 0$ as a telescoping series:

$$\begin{aligned} x-1-\log x &= \lim_{n\to\infty}\sum_{k=1}^{n}\Big[2^{k-1}\big(x^{2^{1-k}}-1\big) - 2^k\big(x^{2^{-k}}-1\big)\Big] \\ &= \sum_{k=1}^{\infty}\Big[2^{k-1}\big(x^{2^{-k}}-1\big)\big(x^{2^{-k}}+1\big) - 2^k\big(x^{2^{-k}}-1\big)\Big] \\ &= \sum_{k=1}^{\infty} 2^{k-1}\big(x^{2^{-k}}-1\big)\Big[\big(x^{2^{-k}}+1\big)-2\Big] = \sum_{k=1}^{\infty} 2^{k-1}\big(x^{2^{-k}}-1\big)^2. \end{aligned} \tag{6}$$

■

Since each term of the series (6) is clearly positive for $0 < x \neq 1$, the inequality $\log x < x - 1$ follows immediately. Moreover, a brief inspection of the operations used to form the terms of the series (6) indicates that the natural logarithm can be calculated in principle at any point of its domain using only the operations of addition, subtraction, multiplication, and square root extraction repeatedly.

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Summary. The natural logarithm can be represented by an infinite series which, in contrast with its Taylor series, converges for all positive real values of the variable, and makes the fundamental property of concavity patently obvious. In principle, the series can be used to calculate the natural logarithm at any point of its domain using only the operations of addition, subtraction, multiplication, and square root extraction repeatedly.

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Could Euler Have Conjectured the Prime Number Theorem?

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Euler likely was not motivated to study the distribution of prime numbers. However, conjecturing the prime number theorem wouldn't have been that hard, based on what he knew, in particular the approximate growth rate of the partial sums $\sum_{p\leq x}\frac{1}{p}$, where p runs over only the primes up to x. Let us take a look at what Euler did do and how he could have pushed it a little further had he been sufficiently interested.

The sum of reciprocals of primes

Euler showed in his paper "Variae observationes circa series infinitas" [**6**, Theorema 19] (Euler Archive number E72), presented to the St. Petersburg Academy in 1737 and published in 1744, that the sum of the reciprocals of the primes diverges, and he even worked out the growth rate of $\sum_{p\leq x}\frac{1}{p}$. As was typical of Euler, his method of discovering this was beautiful, even if his lack of respect for divergent series may look horrifying to a modern mathematician. We present something similar to what Euler did in the remainder of this section; we have taken logarithms of every expression in his paper to be more consistent with modern presentations.

It was already known that the harmonic series grows like $\log x$; that is,

$$\sum_{n=1}^{x}\frac{1}{n}\approx\log(x),$$

or, more precisely,

$$\lim_{x\to\infty}\left(\sum_{n=1}^{x}\frac{1}{n}-\log(x)\right)=\gamma\approx 0.577.$$

Figure 1 shows that $0<\gamma<1$.

Euler also knew a product formula, now known as the Euler product, for what would later be named the Riemann zeta function:

$$\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p\text{ prime}}\left(1-\frac{1}{p^s}\right)^{-1} \tag{1}$$

whenever $\Re(s)>1$. This formula has a delightful interpretation as an analytic formulation of the fundamental theorem of arithmetic; this connection is described in many places, for instance [**1**]. Despite the fact that neither side of (1) makes sense when $s=1$, Euler bravely set $s=1$ to obtain

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots=\left(1-\frac{1}{2}\right)^{-1}\left(1-\frac{1}{3}\right)^{-1}\left(1-\frac{1}{5}\right)^{-1}\left(1-\frac{1}{7}\right)^{-1}\cdots. \tag{2}$$

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MSC: Primary 11N05

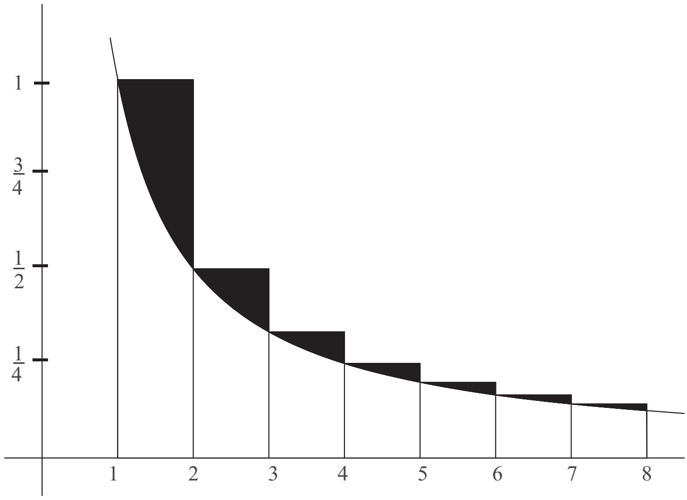

Figure 1 A "proof by picture" showing that $0 < \gamma < 1$.

Formula (2) is Theorema 7 in [**6**]. Taking logarithms, he then obtained

$$\log\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right) = -\log\left(1-\frac{1}{2}\right) - \log\left(1-\frac{1}{3}\right) - \log\left(1-\frac{1}{5}\right) - \log\left(1-\frac{1}{7}\right) - \cdots. \tag{3}$$

Next, he used the power series expansion

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

and regrouped terms so that the right side of (3) becomes

$$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots\right) + \frac{1}{2}\left(\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right) + \frac{1}{3}\left(\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\cdots\right) + \cdots. \tag{4}$$

Since all the groups but the first add up to something finite, they will not make a significant contribution in the limit, and Euler ignored them. Hence, in the limit,

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots = \log\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right) = \log\log\infty.$$

This result is Theorema 19 in [**6**]. In modern notation, we would write

$$\sum_{\substack{p\le x \\ p \text{ prime}}} \frac{1}{p} \approx \log\log x. \tag{5}$$

One can get rid of all the divergent series and odd-looking expressions like $\log\log\infty$ by taking limits and being careful with error terms, as a modern mathematician would do, but that wasn't a necessary part of the mathematical culture in the 18th century.

On to the prime number theorem

At this point, Euler stopped. However, he could have used (5) to estimate the number of primes up to x, at least conjecturally, using a quick and easy computation. Prime numbers were not the focus of his paper [**6**], but he could easily have returned to the topic at some later point had he chosen to do so.

Let us try to estimate the number of primes between x and kx, for some $k > 1$, and let us write $\pi(x)$ for the number of primes up to x. Based on (5), we estimate

$$\sum_{\substack{x<p\leq kx \\ p \text{ prime}}} \frac{1}{p} \approx \log\log(kx) - \log\log(x).$$

If k is only slightly bigger than 1, then each term in the sum on the left is roughly $\frac{1}{x}$, and the number of terms in the sum is $\pi(kx) - \pi(x)$. Hence, we have

$$\log\log(kx) - \log\log(x) \approx \frac{\pi(kx) - \pi(x)}{x}. \tag{6}$$

The goal is now to estimate $\log\log(kx) - \log\log(x)$. Rewriting (6), we have

$$\pi(kx) - \pi(x) \approx x(\log\log(kx) - \log\log(x)) = x\log\left(1 + \frac{\log k}{\log x}\right).$$

Expanding the outer logarithm as a power series and truncating after the first term, we obtain

$$\pi(kx) - \pi(x) \approx \frac{x\log k}{\log x}.$$

Now, choosing $k = 1 + \frac{1}{x}$ and again taking the first term of the power series for $\log(k) = \log\left(1 + \frac{1}{x}\right) \approx \frac{1}{x}$, we get

$$\pi(x+1) - \pi(x) \approx \frac{1}{\log(x)}. \tag{7}$$

Now, we can determine $\pi(x)$ by summing:

$$\pi(x) = \sum_{n=1}^{x-1}(\pi(n+1) - \pi(n)) \approx \sum_{n=2}^{x-1}\frac{1}{\log(n)} \approx \int_2^x \frac{dt}{\log t},$$

which is one form of the prime number theorem. (We change the lower index of the sum and integral to 2 to avoid the problem of division by 0 that occurs when $n = 1$. This does not affect the asymptotics.) A slightly different heuristic for the prime number theorem also coming from the sum of the reciprocals of the primes can be found in Sandifer's March 2006 column of "How Euler Did It" [**12**]. Yet another way of conjecturing the prime number theorem from the series comes from Abel's technique of partial summation, not yet developed in Euler's time, which allows one to compute $\sum_{n\leq x} c_n f(n)$ from the sum $\sum_{n\leq x} c_n$, here used with $c_n = \frac{1}{n}$ if n is prime and 0 otherwise, and $f(n) = n$; see for instance [**2**, §1.3].

Primes and random variables

So, why didn't Euler do this? One can only speculate. Perhaps he was simply not sufficiently interested in the question of prime density; after all, as arguably the most prolific mathematician in history, he clearly had enough other things to occupy his attention! Indeed, his paper [**6**] was not primarily about prime numbers, but rather about infinite series and their relation to infinite products, a subject that he was deeply interested in throughout his life, culminating in such gems as his solution to the Basel problem of evaluating the sum $\sum_{n=1}^{\infty}\frac{1}{n^2}$ [**5**], E41, and the pentagonal number theorem [**7**], E541. Or perhaps he simply had no reason to believe that there would be any large-scale patterns in the primes. But my suspicion is that something like (7) would have felt very strange to an 18^{th}-century mathematician. After all, what can it possibly mean? The left side is either 0 or 1: it's 0 if $x+1$ is composite and 1 if $x+1$ is prime. But the right side is some real number between 0 and 1, which does not "know" anything about prime numbers; rather, it's decaying smoothly. So (7) is nonsense, but, to quote Gilbert and Sullivan [**8**], "oh, what precious nonsense!"

Modern mathematicians understand things like (7) differently. There is no randomness in the primes: A fixed number n is either prime or composite, and no amount of coin flipping or die rolling will ever change that. However, *statistics* about primes behave like statistics about random numbers, with a certain distribution. Imagine building a set S of positive integers as follows: For each integer $n \geq 2$, pick a random number α_n uniformly on [0, 1], and put n into S if $\alpha_n < \frac{1}{\log n}$, with each α_n being chosen independently from all the rest. This model is known as the Cramér model, after Harald Cramér, who wrote several papers, such as [**3**], describing it. This model is elegantly described and analyzed in a recent survey article [**13**] of Soundararajan.

For instance, the Cramér model predicts the twin prime conjecture, which is still open, in spite of recent progress of Goldston–Pintz–Yıldırım [**9**], Zhang [**14**], Maynard [**10**], and the Polymath project [**11**]. The twin prime conjecture states that there are infinitely many positive integers n so that $n-2$ and n are both prime. The argument goes as follows: The probability that $n-2$ and n are both in S is $\frac{1}{\log(n-2)} \times \frac{1}{\log(n)} \geq \frac{1}{\log(n)^2}$. Thus, the expected number of pairs $n-2, n \in S$ is

$$\sum_{n=4}^{\infty}\frac{1}{\log(n-2)\log(n)} \geq \sum_{n=4}^{\infty}\frac{1}{\log(n)^2} = \infty.$$

This approach must be used with care: For example, the same argument as in the preceding paragraph can be used to show that S is expected to contain infinitely many pairs of consecutive numbers, whereas 2 and 3 are the only consecutive primes, so statistics about pairs of consecutive elements of S and pairs of consecutive primes behave completely differently. However, one might be tempted to say that any statistics about S and the primes agree "unless there is an obvious reason that they don't." Of course, this is only a rough heuristic: After all, one person's obvious reason may be another person's deep result.

While the Cramér model is only a heuristic, it is sometimes possible not only to make conjectures but also to prove theorems by taking a probabilistic approach to primes and divisibility. One of the first examples of these theorems was the Erdős–Kac Theorem [**4**], which shows that the number of prime factors of numbers of size roughly n becomes normally distributed as $n \to \infty$, with mean and variance $\log\log(n)$. Since then, the probabilistic method has become a standard technique in analytic number theory, as well as many other areas of mathematics.

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Summary. In this article, we investigate how Euler might have been led to conjecture the prime number theorem, based on what he knew. We also speculate on why he did not do so.

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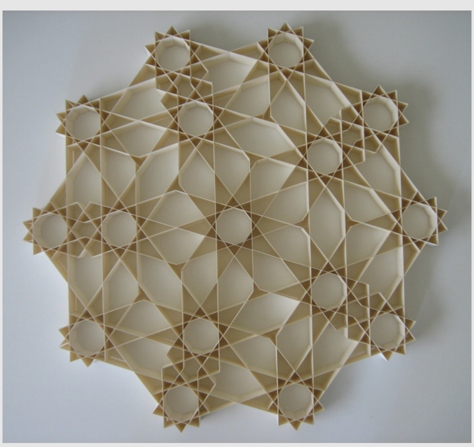

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Chris K. Palmer

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See interview on page 380.

A Dissection Proof of Euler's Series for $1-\gamma$

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Throughout the history of mathematics, there has been much interest in the connection between geometric figures and their areas. As a very early example of this, Archimedes [**8**, p. 233] discovered a method of subdividing a parabolic segment into infinitely many triangles, allowing him to express the area of the region as a geometric series. Since then, other series have been expressed in figures with infinitely subdivided areas. For instance, Lord Brouncker [**2**] partitioned the region below $1/x$ in the interval $[1, 2]$ according to the alternating harmonic series (recently rediscovered in [**9**]), and more recently Viggo Brun [**3, 4**] partitioned a quarter unit circle according to Leibniz's series for $\pi/4$.

In [**11**], both Brouncker's and Brun's partitions were revisited and Taylor series were used to discover new partitions for $\ln 2$ and $\pi/4$. This article demonstrates how the same method may also be applied to the series

$$1-\gamma = \sum_{n=2}^{\infty} \frac{\zeta(n)-1}{n}, \tag{1}$$

where γ is the Euler–Mascheroni constant and $\zeta(n)$ is the Riemann zeta function

$$\zeta(n) := \sum_{k=1}^{\infty} \frac{1}{k^n}.$$

The constant γ, which Euler introduces in [**5**, §6] (his notation for the constant is C), is typically defined as

$$\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right).$$

This constant frequently appears in number theory in the study of prime numbers, as well as in analysis in connection with the Γ function. (See [**7, 12**] for detailed expositions and [**15**] for the origin story.) Interestingly, when Euler calculates the value of γ in [**5**, §11], he does not use its definition directly. Indeed, due to the slow convergence of the harmonic series, this approach would be time-consuming to say the least. Instead, Euler expresses γ as

$$\gamma = \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n}$$

and approximates the value of γ as 0.577218. Over the years, Euler finds several alternative ways of calculating γ, and in 1776 [**6**, §6] he discovers the series (1) for $1-\gamma$. With this series Euler can approximate γ very quickly as 0.5772169 using only 16 terms, which is correct up to the fifth decimal place, since $\gamma = 0.577215\ldots$.

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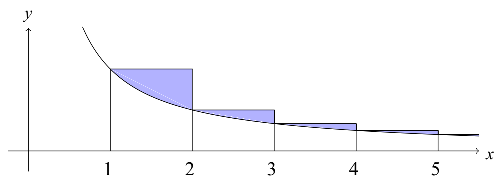
Visualizing γ

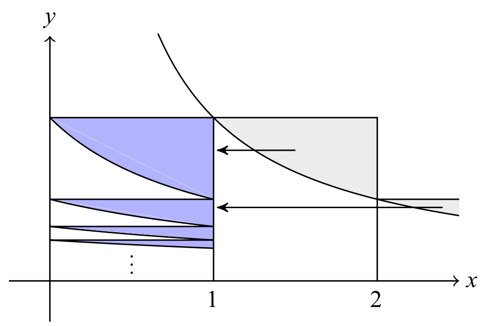
We begin by reproducing a well-known visualization of the Euler–Mascheroni constant γ. (See, for instance, [**1**, p. 345–347],[**13**, p. 95], [**14**].) Interpreting $\ln n$ in the definition of γ as the area under the function $1/x$ in the interval $[1, n]$, we may partition the region according to the intervals $[k, k+1]$, $k = 1, \ldots, n-1$. Thus we may write

$$\gamma = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \int_{k}^{k+1} \frac{dx}{x} \right).$$

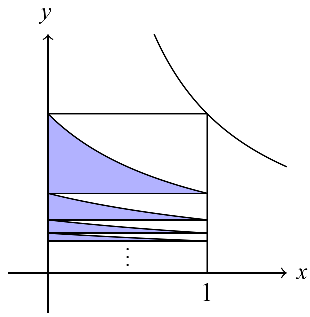
The kth term of this series corresponds to the area between the constant function $1/k$ and the function $1/x$, in the interval $[k, k+1]$.



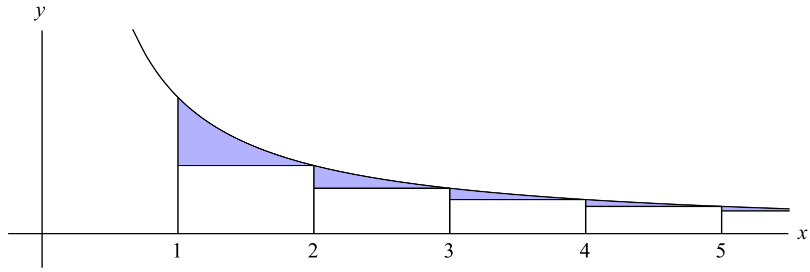
To see visually that this series converges, we imagine sliding each "wedge" to the left against the y-axis.



This figure shows that the combined regions is bounded by the unit square so that the total area is bounded by 1. Thus, we have proven both that the series converges, and that the value of γ is less than 1. For our purposes, the figure also gives us a natural target for Euler's series, which requires a region having area $1 - \gamma$, in the complement of the region representing γ in the unit square.



Recall that the original source of these wedges are the regions bounded above by $1/x$ and below by $1/(k+1)$ in each of the intervals $[k, k+1]$ for $k = 1, 2, 3, \ldots$.



We will call the subregion associated with the interval $[k, k+1]$ the kth wedge, which has area

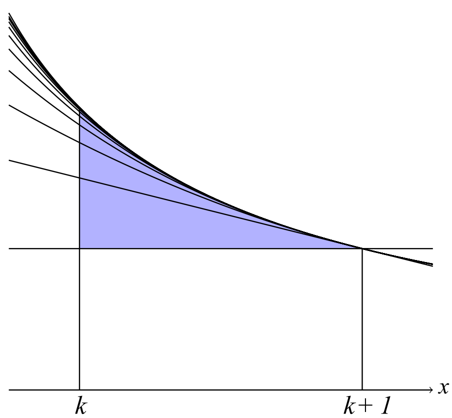
$$A_k = \int_k^{k+1} \frac{dx}{x} - \frac{1}{k+1}, \text{ so that } 1 - \gamma = \sum_{k=1}^{\infty} A_k.$$

Visualizing Euler's series for $1 - \gamma$

We will create our partitions using the Taylor polynomials of the function $1/x$ about $x = k + 1$,

$$T_{kn}(x) = \sum_{r=0}^{n} (-1)^r \frac{(x-(k+1))^r}{(k+1)^{r+1}}.$$

The graphs of the Taylor polynomials in the interval $[k, k+1]$ partition the kth wedge.



To determine the area between consecutive Taylor polynomials, we calculate the integral

$$\begin{aligned} a_{kn} &= \int_k^{k+1} \left(T_{kn}(x) - T_{k(n-1)}(x)\right) dx \\ &= \int_k^{k+1} (-1)^n \frac{(x-(k+1))^n}{(k+1)^{n+1}} dx = \frac{1}{(n+1)(k+1)^{n+1}}. \end{aligned}$$

Thus,

$$A_k = \sum_{n=1}^{\infty} a_{kn} = \sum_{n=1}^{\infty} \frac{1}{(n+1)(k+1)^{n+1}},$$

and it follows that

$$1 - \gamma = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_{kn} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(n+1)(k+1)^{n+1}},$$

which has terms corresponding to known areas. The next step is to interchange summation signs, and then use the definition of the Riemann zeta function. Thus

$$1 - \gamma = \sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{k=1}^{\infty} \frac{1}{(k+1)^{n+1}} = \sum_{n=1}^{\infty} \frac{\zeta(n+1) - 1}{n+1}.$$

Reindexing, we have derived Euler's series

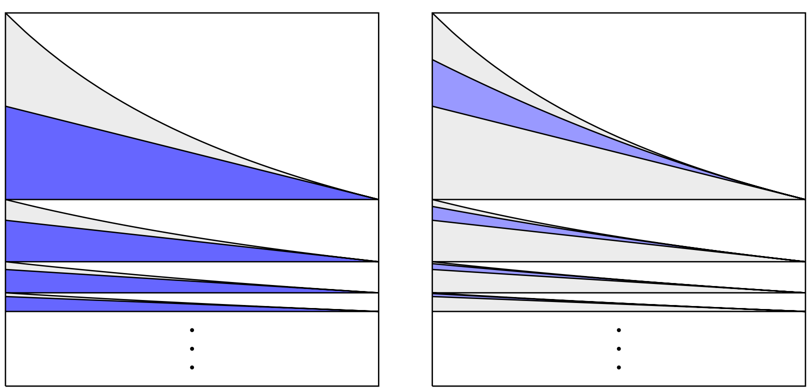
$$1 - \gamma = \sum_{n=2}^{\infty} \frac{\zeta(n) - 1}{n},$$

where each term represents the series

$$\frac{\zeta(n) - 1}{n} = \sum_{k=1}^{\infty} \frac{1}{n(k+1)^n} = \sum_{k=1}^{\infty} a_{k(n-1)}.$$

So what do the terms $(\zeta(n) - 1)/n$ correspond to in our picture? As a result of the interchange of summations, the first ($n = 2$) term corresponds to the areas a_{k1}, $k = 1, 2, 3, \ldots$ in each wedge, namely the regions bounded by the Taylor polynomials T_{k1} and T_{k2}.

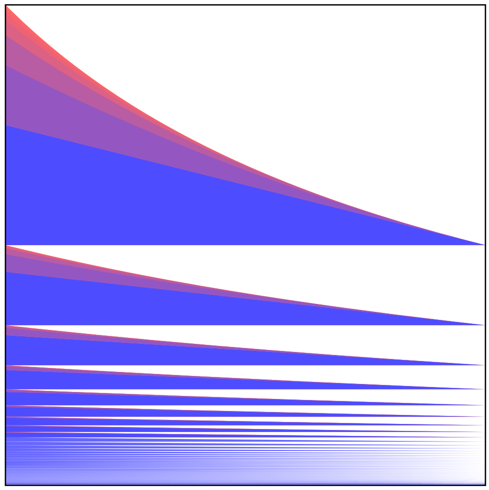
Thus, just as the definition of $1 - \gamma$ may be visualized as a collection of wedges, we may visualize each term of the Euler series as a collection of subwedges. For instance, the first term $(\zeta(2) - 1)/2$ is a collection of triangles, and the second term $(\zeta(3) - 1)/3$ is a collection of subwedges, each bounded by a line and a parabola.



$\frac{\zeta(2) - 1}{2}$ $\frac{\zeta(3) - 1}{3}$

Continuing in this manner, we see that each term will be a collection of subwedges bounded by pairs of polynomials of consecutive degree, with each collection somewhat resembling the parent picture for $1 - \gamma$.

Collecting the pieces above, we arrive at the following partition for the unit square, in which the original series for γ is visible, as well as the Euler series for $1 - \gamma$, which in turn contains the terms of the series for $(\zeta(n) - 1)/n$.



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Summary. We demonstrate a visualization of Euler's series for $1 - \gamma$, where γ is the Euler–Mascheroni constant.

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Colley's Coin: Ranking Sports Teams With Laplace's Rule of Succession

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Controversy erupted in 2000 when Florida State University jumped over the University of Miami in the college football rankings, taking the number 2 spot and giving them a chance at the national title, even though Miami had beaten FSU in head-to-head competition earlier in the season. Florida State's impressive scoring margins contributed to the jump, so in the following season, the NCAA moved to reduce the effect of running up the score by exchanging two of its score-based ranking methods for algorithms that rely only on the outcome of a game: win, lose, or draw. One of these was a linear-algebra-based method devised by Wesley Colley.

Tim Chartier has written nice explanations of the Colley method that highlight its use of linear algebra (see [**2**] and [**1**]), and Colley's own description is enjoyable to read (see [**3**]). At its heart, the Colley method rests on the notion that each team has an intrinsic, quantifiable caliber, or *rating*, R. People generally agree that the simplest way of quantifying a team's caliber based on its performance is

$$R = \frac{W}{T} = \frac{\text{\# wins}}{\text{total \# games played}}, \tag{1}$$

but based on Laplace's rule of succession (used by Laplace to compute the probability that the sun will rise tomorrow, see [**4**] and [**5**]), Colley begins the development of his method with

$$R = \frac{W+1}{T+2}. \tag{2}$$

Many people find this formula unintuitive, especially when compared to equation (1). Colley provides some motivation for the formula by appealing to the metaphor of "locating a marker on a craps table by trial and error shots of dice," and Ross argues the result by applying a limit to a variation on Bayes' law (see [**4**]). The goal of this note is to provide an elementary derivation of equation (2) that is widely accessible and through which we can understand the subtle meaning of the right-hand side. We will combine ideas of conditional probability, integration, and mathematical induction and will rest the argument on the simple notion that two expressions of the same quantity, although they might appear different to the eye, must have equal value.

Preliminary ideas and calculations Recall that a *Bernoulli trial* is an event with only two possible outcomes; in sports, we can treat each head-to-head competition as a Bernoulli trial in which success for a team is defined to be a win. In the context of this note, our goal is to determine a team's rating based on the outcomes of a handful of trials. Subtle assumptions will be lurking in our discussion, including the temporary omission of ties and the idea that the competitions are independent, much like the flipping of a coin. Indeed, the metaphor of a weighted coin is useful in thinking about this topic, so we will use it in the exposition that follows. We will denote by w the

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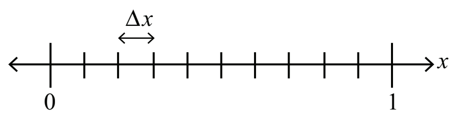
weighting of the coin, by which we mean that w is the probability of seeing "heads," and all $w \in [0, 1]$ will be allowed. The notation $B(n, m+n)$ will mean that we flip our Bernoulli coin $m+n$ times and see n heads. That outcome has a probability of

$$P\Big(B(n, m+n) \Big) = \frac{(m+n)!}{m!\,n!}\, w^n (1-w)^m \tag{3}$$

when w is the weighting of the coin.

Equation (3) allows us to compute the probability of seeing $B(n, m+n)$ when we know w precisely. On the other end of the spectrum, we can also investigate the probability of seeing $B(n, m+n)$ when we know nothing of w beyond the fact that $w \in [0, 1]$. In this case, let's segment $[0, 1]$ into N subintervals of length $\Delta x = 1/N$ and think about what would happen if w were in each.



More specifically, let $I_1 = \left[0, \frac{1}{N}\right]$, and $I_k = \left(\frac{(k-1)}{N}, \frac{k}{N}\right]$ when $1 < k \le N$. Without foreknowledge to tell us differently, we assume that w is equally likely to be in any of the N subintervals, so $\frac{1}{N}$ is the probability that w is in any particular subinterval. Because these N cases ($w \in I_1, w \in I_2, \ldots, w \in I_n$) are jointly exhaustive and mutually exclusive,

$$\begin{aligned} P\Big(B(n, m+n) \Big) &= \sum_{k=1}^{N} P\Big(B(n, m+n) \text{ and } w \in I_k\Big) \\ &= \sum_{k=1}^{N} P\Big(B(n, m+n) \;\Big|\; w \in I_k\Big) P\big(w \in I_k \big), \end{aligned} \tag{4}$$

where $P(Y|X)$ denotes the conditional probability of Y, given X. Now suppose that $\Delta x = 1/N$ is very small so that w is very close to the right end point of I_k, which we'll denote by x_k. In this case, we can use equation (3) to approximate

$$P\Big(B(n, m+n) \;\Big|\; w \in I_k\Big) \approx \frac{(m+n)!}{m!\,n!}\, x_k^n (1-x_k)^m .$$

In combination with the fact that $P(w \in I_k) = \frac{1}{N} = \Delta x$, this approximation allows us to recast equation (4) as

$$P\Big(B(n, m+n) \Big) \approx \sum_{k=1}^{N} \frac{(m+n)!}{m!\,n!}\, x_k^n (1-x_k)^m \, \Delta x ,$$

in which we see the form of a Riemann sum. The error in this approximation vanishes in the limit as $N \to \infty$, and we have

$$P\Big(B(n, m+n)\Big) = \frac{(m+n)!}{m!\,n!} \int_0^1 x^n (1-x)^m \, dx . \tag{5}$$

In the Addendum to this note, we employ mathematical induction to establish a closed-form expression for the right-hand side of equation (5), which we cite here for convenience:

$$P\Big(B(n, m+n)\Big) = \frac{1}{m+n+1} . \tag{6}$$

A cumulative distribution function Equation (3) rests on the assumption that we know w exactly, and equation (6) assumes that we know nothing about w. We're interested in a scenario that is somewhere between these two extremes because the very occurrence of $B(n, m+n)$ constitutes *some* information about the coin. So let us turn our attention to this "in between" question. What is the probability that $w \in [0, p]$ *and* we see $B(n,\ m+n)$? Here again the idea of conditional probability is helpful:

$$P\Big(w \in [0, p] \text{ and } B(n, m+n)\Big) = P\Big(w \in [0, p]\Big) \times P\Big(B(n, m+n) \;\Big|\; w \in [0, p]\Big). \tag{7}$$

On the right-hand side of equation (7), we see a product of two factors. The first of these is the probability that $w \in [0, p]$, which is just p (the fraction of $[0, 1]$ taken by $[0, p]$). To formulate the second factor, we need only a minor adjustment in the argument that led to equation (5). When we subdivide $[0, p]$ into N subintervals of length $\Delta x = \frac{p}{N}$, which we'll denote by $\tilde{I}_1, \tilde{I}_2, \dots, \tilde{I}_N$, the probability that w is in any particular subinterval is $\frac{1}{N}$, so

$$\begin{aligned} P\Big(B(n, m+n) \;\Big|\; w \in [0, p]\Big) &= \sum_{k=1}^{N} P\Big(B(n, m+n) \;\Big|\; w \in \tilde{I}_k\Big) \times P\Big(w \in \tilde{I}_k\Big) \\ &\approx \sum_{k=1}^{N} \frac{(m+n)!}{m!\,n!}\, x_k^n (1-x_k)^m \left(\frac{1}{N}\right), \end{aligned}$$

where x_k is the right endpoint of $\tilde{I}_k$. Then because $\frac{1}{N} = \frac{\Delta x}{p}$,

$$P\Big(B(n, m+n) \;\Big|\; w \in [0, p]\Big) \approx \sum_{k=1}^{N} \frac{(m+n)!}{m!\,n!}\, x_k^n (1-x_k)^m \,\frac{\Delta x}{p}.$$

In the limit as $N \to \infty$, this yields

$$P\Big(B(n, m+n) \;\Big|\; w \in [0, p]\Big) = \frac{(m+n)!}{m!\,n!\,p} \int_0^p x^n (1-x)^m \, dx$$

so that equation (7) becomes

$$P\Big(w \in [0, p] \text{ and } B(n, m+n)\Big) = \frac{(m+n)!}{m!\,n!} \int_0^p x^n (1-x)^m \, dx. \tag{8}$$

In equation (7), we displayed one way of calculating the probability that both $w \in [0, p]$ and $B(n, m+n)$, and this led us to equation (8). Calculating the same probability in a different way is the key step toward understanding equation (2):

$$P\Big(w \in [0, p] \text{ and } B(n, m+n)\Big) = P\Big(B(n,\ m+n)\Big) \times P\Big(w \in [0, p] \;\Big|\; B(n, m+n)\Big).$$

Using equation (6), we can rewrite this as

$$P\Big(w \in [0, p] \text{ and } B(n, m+n)\Big) = \left(\frac{1}{n+m+1}\right) \times P\Big(w \in [0, p] \;\Big|\; B(n, m+n)\Big). \tag{9}$$

The right-hand sides of equation (8) and (9) are two different ways of calculating the same probability, so they are equal to each other. That is,

$$\left(\frac{1}{n+m+1}\right) \times P\Big(w \in [0, p] \;\Big|\; B(n, m+n)\Big) = \frac{(m+n)!}{m!\,n!} \int_0^p x^n (1-x)^m \, dx \, .$$

After multiplying both sides of this equation by $(n+m+1)$, we arrive at

$$P\Big(w \in [0, p] \;\Big|\; B(n, m+n)\Big) = \frac{(m+n)!\,(n+m+1)}{m!\,n!} \int_0^p x^n (1-x)^m \, dx \, .$$

This function of p is a cumulative distribution function, and we will denote it by $\Phi(p)$. Note that Φ incorporates the fact that $B(n,\ m+n)$ has occurred! You can see important features of Φ in Figure 1, which shows a graph of Φ in a particular instance. Specifically,

$\Phi(1) = 1$, which makes sense because w is certainly somewhere in $[0, 1]$

$\Phi(0) = 0$, which makes sense because $[0, 0]$ has no width, so the probability of finding $w \in [0, 0]$ "should be" 0

$\Phi'(p) > 0$, which makes sense because increasing the width of $[0, p]$ makes it more likely that $w \in [0, p]$, so Φ "should be" increasing.

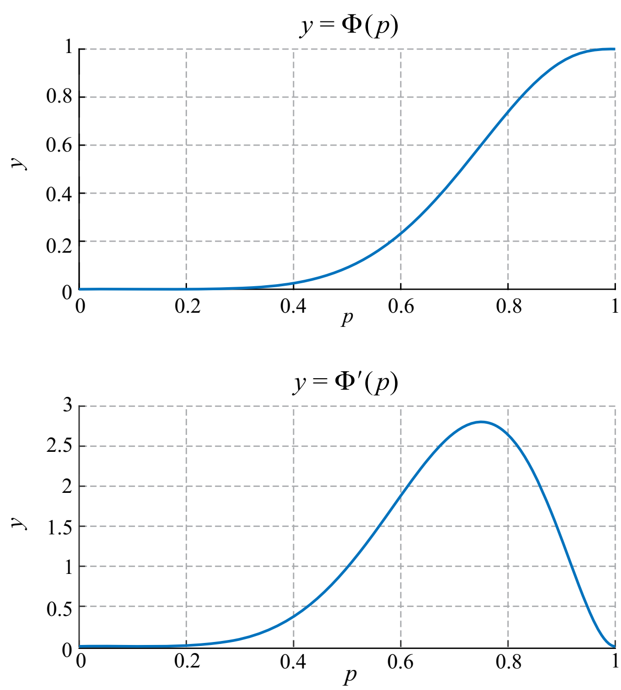


Figure 1 (both) Six success in eight Bernoulli trials ($n = 6$ and $m = 2$); (top) the graph of Φ; (bottom) the graph of Φ' (vertical scale not the same).

Expected value The fundamental theorem of calculus tells us that the probability density function associated with Φ is

$$\Phi'(x) = \frac{(m+n)!\,(n+m+1)}{m!\,n!} x^n (1-x)^m ,$$

and with this function we can compute the expected value of w:

$$\mathcal{E} = \int_0^1 x\,\Phi'(x)\,dx = \frac{(m+n)!\,(n+m+1)}{m!\,n!} \int_0^1 x^{n+1}(1-x)^m\,dx .$$

After using equation (6) to evaluate the integral, we see that

$$\mathcal{E} = \frac{(m+n)!\,(n+m+1)}{m!\,n!} \frac{m!(n+1)!}{(m+(n+1)+1)!} = \frac{n+1}{m+n+2}. \tag{10}$$

Interpretation Now let's interpret this result in the context of head-to-head competitions. If a team plays a total of $T = m + n$ games and wins $W = n$ of them, equation (10) tells us that the expected probability of success is

$$\frac{W+1}{T+2} ,$$

which is the right-hand side of equation (2). Loosely said, equation (2) suggests that a team's *rating* is the probability of success in the next trial (game) as calculated based on prior outcomes. More accurately, it's the expected value of that probability.

At the beginning of this note, we mentioned some assumptions that led us to treat a team's games as coin flips, but something else affects the outcome: the other team! So Colley adapted equation (2) to incorporate the caliber of opponents in the calculation of a team's rating, thereby including "strength of schedule" in the ranking decision. The adaptation is clever, but that's a story for another day.

Addendum

In equation (6), we asserted that

$$\frac{(m+n)!}{m!\,n!} \int_0^1 x^n (1-x)^m\,dx = \frac{1}{m+n+1}.$$

The case when $m = 1$ is straightforward. For all natural numbers n,

$$\int_0^1 x^n (1-x)^1\,dx = \int_0^1 (x^n - x^{n+1})\,dx = \frac{n!}{(n+2)!} . \tag{11}$$

Integration by parts allows us to reduce the case of $m = 2$ to the case of $m = 1$, which we computed above. Specifically,

$$\begin{aligned}\int_0^1 x^n (1-x)^2\,dx &= \frac{1}{n+1} x^{n+1}(1-x)^2 \Big|_{x=0}^{x=1} + \int_0^1 \frac{2}{n+1} x^{n+1}(1-x)^1\,dx \\ &= (0-0) + \left(\frac{2}{n+1}\right) \int_0^1 x^{n+1}(1-x)^1\,dx .\end{aligned}$$

Then by writing $N = n + 1$ and using equation (11), we see

$$\left(\frac{2}{N}\right)\int_0^1 x^N(1-x)^1\,dx = \left(\frac{2}{N}\right)\frac{N!}{(N+2)!} = \frac{(N-1)!(2)}{(N+2)!} = \frac{n!(2)}{(n+2+1)!}.$$

If you don't yet see the pattern, try using the same technique with $m = 3$.

The pattern that seems to be emerging from $m = 1, 2$ and 3 leads us to conjecture that

$$\int_0^1 x^n(1-x)^m\,dx = \frac{n!m!}{(n+m+1)!}.$$

Based on our experience using integration by parts to establish the result at one value of m by relating it to a previous result, it seems natural to use mathematical induction to prove that our conjecture is correct. We have already established a base case ($m = 1, 2, 3$), so let's progress to the inductive step. Our premise is that there is a value of k for which

$$\int_0^1 x^n(1-x)^k\,dx = \frac{n!k!}{(n+k+1)!}$$

has been verified for every positive integer n.

In our investigation of $m = k + 1$, integration by parts allows us to see that

$$\int_0^1 x^n(1-x)^{k+1}\,dx = \left(\frac{k+1}{n+1}\right)\int_0^1 x^{n+1}(1-x)^k\,dx,$$

after which the induction premise allows us to evaluate the remaining integral:

$$\left(\frac{k+1}{n+1}\right)\int_0^1 x^{n+1}(1-x)^k\,dx = \frac{n!(k+1)!}{(n+(k+1)+1)!} = \frac{n!\,m!}{(n+m+1)!}.$$

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Summary. The Colley ranking method was adopted by the NCAA to rank college football teams and has also been used by ecologists and social scientists. This ranking method is based on the ratio (wins+1)/(games played+2). At first glance, this ratio appears close enough to the familiar (wins)/(games played) ratio that many people feel it should be easily understood, and yet most people find it strikingly unintuitive. In this sense, the ratio is in a kind of mathematical "zombie zone"—simple enough that it should be intuitive and yet not intuitive. In this note, we provide an elementary derivation of the ratio that is accessible to undergraduates and ties together three disparate parts of the curriculum: integration by parts, mathematical induction, and probability.

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On Indeterminate Forms of Exponential Type

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A common topic in calculus is to use L'Hospital's rule to evaluate limits involving indeterminate forms of exponential type (e.g., $\lim_{x\to x_0} f(x)^{g(x)}$). In this note, we consider simple methods to evaluate such limits that do not require L'Hospital's rule, useful, for example, when the functions are not differentiable.

The following authors have used L'Hospital's rule to the transformation $g(x)\ln f(x)$ to determine when $\lim_{x\to 0} f(x)^{g(x)} = 1$ for the limits of f and g both approaching 0. In [**2**], Paige proved that $\lim_{x\to 0^+} x^{f(x)} = 1$ holds for f with $f(0) = 0$ that possess a derivative in a neighborhood of the origin. Motivated by this result, Rotando and Korn [**3**] showed that if f and g are real functions that vanish at the origin and are analytic at 0 (infinitely differentiable is not sufficient), then $\lim_{x\to 0^+} f(x)^{g(x)} = 1$. The same conclusion was deduced by Baxley and Hayashi [**1**], who also obtained a more general result, in which no smoothness conditions are required (see Corollary 1).

We apply the squeeze theorem below to provide conditions for when $\lim_{x\to x_0} f(x)^{g(x)} = 1$ given that $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$. We denote such a case as 0^0, and extend the notation for other cases such as ∞^0. Perhaps surprisingly, the result follows only from the boundedness of the quotient $\frac{|g(x)|^\mu}{f(x)^\lambda}$ for some constants μ and λ. Hence, we do not require f and g to be differentiable as we do not apply L'Hospital's rule.

Theorem 1. (0^0) *Assume that* $\lim\limits_{x\to x_0} g(x) = 0$ *and* $f(x)$ *is positive in a deleted neighborhood of* x_0 *satisfying* $\lim\limits_{x\to x_0} f(x) = 0$. *If there exist* $\mu > 0$ *and* $\lambda > 0$ *such that* $\frac{|g(x)|^\mu}{f(x)^\lambda}$ *is bounded in a deleted neighborhood of* x_0, *then* $\lim\limits_{x\to x_0} f(x)^{g(x)} = 1$.

Proof. By the Lagrange mean value theorem, we have $(\ln x - \ln 1)/(x-1) = 1/\xi$ where ξ lies between 1 and x. Since $(x-1)/x \le (x-1)/\xi \le x-1$, then $1 - 1/x \le \ln x \le x - 1$. Replacing x by $f(x)^{\frac{\lambda}{\mu+1}}$ we obtain

$$1 - \frac{1}{f(x)^{\frac{\lambda}{\mu+1}}} \le \ln\left[f(x)^{\frac{\lambda}{\mu+1}}\right] \le f(x)^{\frac{\lambda}{\mu+1}} - 1. \tag{1}$$

In order to show this theorem, it suffices to prove that $\lim\limits_{x\to x_0} g(x)\ln f(x) = 0$, i.e.,

$$\lim_{x\to x_0} \frac{\mu+1}{\lambda} g(x) \ln\left[f(x)^{\frac{\lambda}{\mu+1}}\right] = 0. \tag{2}$$

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Obviously we have $\lim_{x\to x_0} g(x)[f(x)^{\frac{\lambda}{\mu+1}} - 1] = 0$. By (1) and the squeeze theorem, if

$$\lim_{x\to x_0} g(x)\left(1 - \frac{1}{f(x)^{\frac{\lambda}{\mu+1}}}\right) = 0, \tag{3}$$

then (2) holds. Now let us turn to prove (3). From the hypothesis we have

$$\lim_{x\to x_0}\left(|g(x)|\frac{1}{f(x)^{\frac{\lambda}{\mu+1}}}\right)^{\mu+1} = \lim_{x\to x_0}|g(x)|\frac{|g(x)|^{\mu}}{f(x)^{\lambda}} = 0,$$

which implies that $\lim_{x\to x_0} g(x)\frac{1}{f(x)^{\frac{\lambda}{\mu+1}}} = 0$, and hence,

$$\lim_{x\to x_0} g(x)\left(1 - \frac{1}{f(x)^{\frac{\lambda}{\mu+1}}}\right) = \lim_{x\to x_0} g(x) - \lim_{x\to x_0} g(x)\frac{1}{f(x)^{\frac{\lambda}{\mu+1}}} = 0.$$

This completes the proof of Theorem 1. ■

If $\lim_{x\to x_0} g(x) = 0$ and $\lim_{x\to x_0} f(x) = +\infty$, then $\lim_{x\to x_0} g(x)[f(x)^{\frac{\lambda}{\mu+1}} - 1] = 0$ holds for $\mu > 0$ and $\lambda < 0$. A similar proof to Theorem 1 can be used to prove the case ∞^0.

Theorem 2. **(∞^0)** *Assume that* $\lim_{x\to x_0} g(x) = 0$ *and* $\lim_{x\to x_0} f(x) = +\infty$. *If there exist* $\mu > 0$ *and* $\lambda < 0$ *such that* $\frac{|g(x)|^{\mu}}{f(x)^{\lambda}}$ *is bounded in a deleted neighborhood of* x_0, *then* $\lim_{x\to x_0} f(x)^{g(x)} = 1$.

We remark here that Theorems 1 and 2 remain true if the condition "$\frac{|g(x)|^{\mu}}{f(x)^{\lambda}}$ is bounded in a deleted neighborhood of x_0" is replaced by "$\lim_{x\to x_0} \frac{|g(x)|^{\mu}}{f(x)^{\lambda}}$ is finite." These two theorems include the work of Baxley and Hayashi [**1**], which is stated as follows.

Corollary 1. **(0^0, ∞^0)** *Suppose* $\lim_{x\to 0^+} g(x) = 0$, *if there exists a real number* α *such that* $b(x) = f(x)/x^{\alpha}$ *is positive bounded, and bounded away from* 0 *as* $x \to 0^+$, *and* $g(x) = x^{\beta}c(x)$ *where* $c(x)$ *is bounded and* $\beta > 0$, *then* $\lim_{x\to 0^+} f(x)^{g(x)} = 1$.

Note that when $\alpha > 0$ the indeterminate form is 0^0, and when $\alpha < 0$ the indeterminate form is ∞^0. This is a corollary to Theorems 1 and 2 since $\frac{|g(x)|^{\mu}}{f(x)^{\lambda}} = \frac{c(x)x^{\beta\mu}}{b(x)x^{\alpha\lambda}}$ is finite when $\lambda = 1/\alpha$ and $\mu = 1/\beta$. The next theorem considers the case 1^{∞}.

Theorem 3. **(1^{∞})** *Assume that* $\lim_{x\to x_0} f(x) = 1$ *and* $\lim_{x\to x_0} g(x) = \infty$. *If* $\lim_{x\to x_0}[f(x) - 1]g(x) = \alpha$, *then* $\lim_{x\to x_0} f(x)^{g(x)} = e^{\alpha}$.

Proof. We have from (1) that $g(x)\big[1 - \frac{1}{f(x)}\big] \le g(x)\ln f(x) \le g(x)[f(x) - 1]$. By the hypothesis we get

$$\begin{aligned}\lim_{x\to x_0} g(x)\Big[1 - \frac{1}{f(x)}\Big] &= \lim_{x\to x_0} g(x)[f(x) - 1]\lim_{x\to x_0}\frac{1}{f(x)} \\ &= \lim_{x\to x_0} g(x)[f(x) - 1] = \alpha,\end{aligned}$$

from which we deduce the theorem. ■

The corollary below is a slight modification of Theorem 3.

Corollary 2. ($(1+0)^\infty$) *Assume that* $\lim_{x\to x_0} f(x) = 0$ *and* $\lim_{x\to x_0} g(x) = \infty$. *If* $\lim_{x\to x_0} f(x)g(x) = \alpha$, *then* $\lim_{x\to x_0} [1 + f(x)]^{g(x)} = e^\alpha$.

TABLE 1: Results proved by squeeze theorem

Type	Condition	Result
0^0	○ $\lim_{x\to x_0} f(x) = 0,\ \lim_{x\to x_0} g(x) = 0$ • $\frac{\lvert g(x)\rvert^\mu}{f(x)^\lambda}$ is bounded for some $\mu > 0, \lambda > 0$ in $U^o(x_0)$	$\lim_{x\to x_0} f(x)^{g(x)} = 1$
∞^0	○ $\lim_{x\to x_0} f(x) = \infty,\ \lim_{x\to x_0} g(x) = 0$ • $\frac{\lvert g(x)\rvert^\mu}{f(x)^\lambda}$ is bounded for some $\mu > 0, \lambda < 0$ in $U^o(x_0)$	$\lim_{x\to x_0} f(x)^{g(x)} = 1$
1^∞	○ $\lim_{x\to x_0} f(x) = 1,\ \lim_{x\to x_0} g(x) = \infty$ • $\lim_{x\to x_0} [f(x) - 1]g(x) = \alpha$	$\lim_{x\to x_0} f(x)^{g(x)} = e^\alpha$
$(1+0)^\infty$	○ $\lim_{x\to x_0} f(x) = 0,\ \lim_{x\to x_0} g(x) = \infty$ • $\lim_{x\to x_0} f(x)g(x) = \alpha$	$\lim_{x\to x_0} [1 + f(x)]^{g(x)} = e^\alpha$

We have established conditions which allow the calculation of the indeterminate form of exponential type by the squeeze theorem instead of L'Hospital's rule and do not require the differentiability of the functions involved. Table 1 lists our results. The corresponding results for x approaching x_0^+, x_0^- or $\pm\infty$ are obtained by using the appropriate change of variables.

Finally, we end this note by giving the following example to illustrate our method.

Example 1. (0^0) We show that $\lim_{x\to 0^+} \left(x \ln \frac{1}{x}\right)^{x^2} = 1$. Since $\lim_{x\to 0^+} x \ln \frac{1}{x} = \lim_{x\to 0^+} x^2 = 0$ and

$$\lim_{x\to 0^+} \frac{|x^2|^\mu}{\left(x \ln \frac{1}{x}\right)^\lambda} = \lim_{x\to 0^+} \frac{x}{x \ln \frac{1}{x}} = \lim_{x\to 0^+} \frac{1}{\ln \frac{1}{x}} = 0,$$

where we have chosen $\mu = \frac{1}{2}$ and $\lambda = 1$, then the result follows by Theorem 1.

One can check this result by using L'Hospital's rule. However, we cannot use Corollary 1 because it is impossible to find a bounded function $b(x) = [x \ln \frac{1}{x}]/x^\alpha$ bounded away from 0 as $x \to 0^+$.

Our method cannot deal with all limits 0^0 and ∞^0 which have value 1, especially whose base functions are exponential. A nice counterexample is given by $\lim_{x\to 0^+} (e^{-1/x})^{x^2} = 1$; our technique fails since for any $\mu, \lambda > 0$,

$$\lim_{x\to 0^+} \frac{x^{2\mu}}{e^{-\lambda/x}} = \lim_{x\to +\infty} \frac{x^{-2\mu}}{e^{-\lambda x}} = \lim_{x\to +\infty} \frac{e^{\lambda x}}{x^{2\mu}} = +\infty.$$

Acknowledgment The authors are indebted to the editor Michael Jones and the referees for useful comments and suggestions, which greatly improved the quality of this note. Jinsen Xiao is supported by the Training Project for Young Teachers Project of Guangdong (Grant No. YQ2015117). Jianxun He is supported by the Speciality Comprehensive Reform Project of Guangdong.

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Summary. We apply the squeeze theorem, instead of L'Hospital's rule, to evaluate limits in indeterminate form of exponential type. We do not require differentiability, instead needing the boundedness of a quotient.

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O	B	I	T		M	I	N	C	E		M	O	S	T
I	L	S	A		C	R	E	A	M		A	R	E	A
N	A	A	N		V	O	I	L	A		T	A	R	E
K	N	I	T	T	I	N	G	C	I	R	C	L	E	
	C	D	O	T		S	H	I	L	O	H			
			O	O	O					O	I	L	E	R
L	A	B		P	R	O	J	E	C	T	N	E	X	T
I	C	E	D		C	R	E	P	E		G	A	P	E
R	A	D	I	C	A	L	D	A	S	H		D	O	S
R	A	E	S	Z					T	I	M			
			C	A	P	U	T	O		D	A	D	E	
	G	E	R	R	Y	M	A	N	D	E	R	I	N	G
S	O	M	E		R	A	T	I	O		I	E	E	E
A	B	I	T		E	M	A	C	S		O	G	R	E
N	I	L	E		S	I	R	E	E		N	O	O	K

The Binomial Theorem Procured From the Solution of an ODE

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We give an alternate proof of the binomial theorem by solving an nth order linear nonhomogeneous ordinary differential equation with constant coefficients. The method used is standard and can be found in any introductory book on ordinary differential equations (for example, see [**5**, pp. 111–118]).

Theorem 1. (Binomial Theorem) *Let n be any nonnegative integer and x, y two real numbers. Then*

$$(x+y)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k y^{n-k}, \tag{1}$$

where 0^0 is interpreted as unity whenever $x = 0$ or $y = 0$.

Proof. For $x = 0$ or $y = 0$, (1) clearly holds. Assume $y \neq 0$ and let $t = x/y$. It is sufficient to show

$$(1+t)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} t^k.$$

Consider the following nth order linear nonhomogeneous ordinary differential equation:

$$f^{(n)}(t) = n!, \tag{2}$$

with initial conditions

$$f^{(k)}(0) = \frac{n!}{(n-k)!}, \quad k = 0, 1, \ldots, n-1. \tag{3}$$

Integrating (2) step by step and simultaneously using (3) to determine the constants obtained during this process, we get $f(t) = (1+t)^n$ as the solution of the above differential equation. The solution of (2) can alternatively be obtained as follows:

Let $f_h(t)$ be the general solution for the homogeneous part of the given differential equation. Substituting $f(t) = e^{rt}$ in $f^{(n)}(t) = 0$, we get the corresponding characteristic equation $r^n = 0$ which has 0 as the root with multiplicity n. Therefore, $f_h(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}$, where $c_0, c_1, \ldots, c_{n-1}$ are arbitrary constants. Also, it is easy to see that $f_p(t) = t^n$ is the particular integral of (2). Hence, the solution is given by

$$f(t) = f_h(t) + f_p(t) = \sum_{k=0}^{n-1} c_k t^k + t^n. \tag{4}$$

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MSC: Primary 34A99; Secondary 05A10

From the initial conditions (3), we get $c_k = n!/k!(n-k)!$, $k = 0, 1, \ldots, n-1$. On substituting the values of c_k's in (4), we obtain

$$f(t) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} t^k. \tag{5}$$

By uniqueness of the solution of differential equation (2), the proof follows. ■

Remark. Note that (5) can alternatively be obtained as follows. On taking the Laplace transform on both sides of (2), we get

$$\mathcal{L}\{f^{(n)}(t)\}(s) = \mathcal{L}\{n!\}(s)$$

$$s^n \mathcal{L}\{f(t)\}(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) = \frac{n!}{s}$$

$$\mathcal{L}\{f(t)\}(s) = \sum_{k=0}^{n} \frac{n!}{(n-k)!} \frac{1}{s^{k+1}}.$$

Taking the inverse Laplace transform, we obtain

$$f(t) = \sum_{k=0}^{n} \frac{n!}{(n-k)!} \mathcal{L}^{-1}\left\{\frac{1}{s^{k+1}}\right\}(t) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} t^k.$$

The binomial theorem is an important mathematical result that expands $(x+y)^n$, for integer $n \geq 0$, into the sum of the products of integer powers of real numbers x and y. It has many applications in different areas of mathematics including statistics and computing. Several proofs of the binomial theorem are available in literature and are either based on the principle of mathematical induction (see [**1**, pp. 59–60]; [**2**]; [**7**, pp. 8–9]) or on counting arguments (see [**7**, p. 9]). Rosalsky [**6**] provided a probabilistic proof of the binomial theorem using the binomial distribution, whereas Hwang [**3**] proved it using differential calculus. Recently, Kataria [**4**] obtained an alternative proof using Laplace transform.

Since elementary probability and counting arguments are used in the construction of the binomial distribution, the proof by Rosalsky [**6**] is not completely independent of the combinatorial proof. Contrary to the proofs involving mathematical induction, the exact form of the binomial coefficients in the proofs due to Ross [**7**, p. 9], Rosalsky [**6**], and Hwang [**3**] does not need to be known in advance. But even in these proofs the following expansion is deduced based on some counting arguments:

$$(x+y)^n = \sum_{k=0}^{n} C(n,k) x^k y^{n-k},$$

where $C(n,k)$'s are nonnegative integers. Our proof of the binomial theorem does not use any such forms and is thus completely independent of the combinatorial proof. Although our proof may not be economical, a student familiar with basic concepts of ordinary differential equations may find it simpler and interesting.

Acknowledgment The author acknowledges the financial support of a research fellowship by UGC, Govt. of India under the Grant F.2-2/98(SA-I).

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Summary. We obtain the binomial theorem as a unique solution of an nth order linear nonhomogeneous ordinary differential equation with constant coefficients and given initial conditions.

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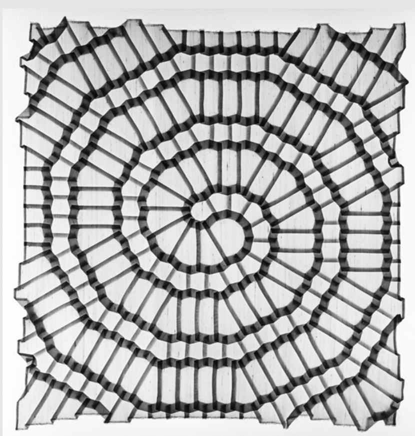

Artist Spotlight:
Chris K. Palmer

Voderburg tiling, © Chris K. Palmer; silk, c. 1997. Voderburg tiling interpreted as a pleated silk Shadowfold. Image courtesy of the artist.

See interview on page 380.

ACROSS

1. Life lines?
5. Chop finely
10. As of 2017, it's known that the Ramsey number $R(5)$ is at ___ 48
14. Ingrid's *Casablanca* role
15. 60s rock band with Eric Clapton on guitar
16. Length times width, for a rectangle
17. Indian bread
18. "Behold!"
19. Set to zero, as a scale
20. * JMM evening social activity "centered" around a certain hobby
23. LaTeX command for a multiplication symbol
24. Major battle of the American Civil War (and, ironically, a Hebrew word meaning "place of peace")
25. Elongated outcry of despair
27. Hockey player in Edmonton
31. Programming language: MAT___
34. * MAA group sponsoring a JMM invited address on Teaching and Learning, given by Stanford's Jo Boaler
39. ___ tea
41. Thin pancake
42. Open wide
43. * JMM multi-day scavenger hunt for students
46. Windows forerunner
47. Eponym of a representation theorem about a Hilbert space and its dual
48. Fields Medalist Gowers (for short)
50. Prison warden Joe on "Orange is the New Black"
55. Miami-___ County
58. * Political subject of a JMM invited address by metric geometer Moon Duchin of Tufts
62. Not much
63. Proportion
64. World's largest assoc. of technical professionals
65. Not much
66. GNU text editor
67. Shrek, e.g.
68. Cairo's river
69. It's said after "yes" or "no", for emphasis
70. Cozy spot

DOWN

1. Farm sound
2. Mel who voiced Bugs Bunny, Pepe Le Pew, and many more
3. "Was It Something ___?": BBC comedy panel game show
4. * MAA's "Mathematician at Large" James giving the JMM Lecture for Students
5. Its prime factorization is II·VII·LXIX
6. Actor Jeremy who played G.H. Hardy in a recent film
7. Farm sound
8. Course that covers limits and derivatives
9. Online communication
10. Hall's Marriage Theorem characterizes whether a bipartite graph has a perfect one
11. Possible format of a qualifying exam
12. Bone-dry
13. ___ kwon do
21. Sporty car roof
22. It could be a square or cube, perhaps
26. Shamu, for one
28. Pb
29. * Another term for JMM's Exhibition Hall
30. Some highways: Abbr.
31. You can take it from Penn Sta. to Montauk
32. Popular juice berry nowadays
33. Saint ___, known as "The Father of English History"
35. NBA's Magic, on scoreboards
36. Nickname of President Bartlet on "The West Wing"
37. Gov. agency currently directed by Scott Pruitt
38. "___ la vie"
40. Real analysis would not be considered this kind of math
44. Old Russian emperor
45. Conceal
49. Mathematician and educator Walter, co-author of "The Art of Problem Posing"
51. Cremation structures
52. Savory taste
53. Speaker of Turkic in Europe and Asia
54. Another way to say 39-Across
56. * With 62-Down, JMM location
57. * Month in which JMM occurs, in a language spoken in and south of 62-Down 56-Down.
58. Mongolian desert
59. Mathematician Artin (but not Borel!)
60. One teaspoon, maybe
61. Many a mathematician
62. * With 56-Down, JMM location

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Joint Mathematics Meetings 2018

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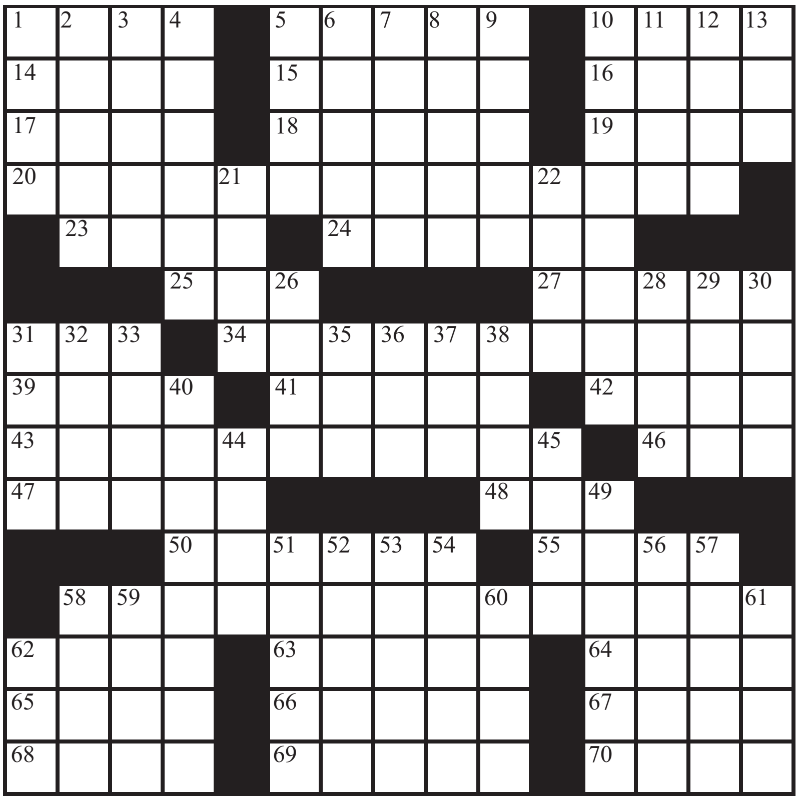

Clues start at left, on page 378. The Solution is on page 374.

Extra copies of the puzzle can be found at the MAGAZINE's website, `www.maa.org/mathmag/supplements`.

Crossword Puzzle Creators

If you are interested in submitting a mathematically themed crossword puzzle for possible inclusion in MATHEMATICS MAGAZINE, please contact the editor at mathmag@maa.org.

Chris K. Palmer: Origami in Action*

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Chris K. Palmer is an artist and currently is the digital fabrication lab manager at UC Berkeley in the College of Environmental Design. Chris is coauthor of the beautifully illustrated book *Shadowfolds* (2011) and is one of the featured artists in the documentary *Between the Folds* (Gould, 2008). We interviewed Chris in Boulder, Colorado at the MoSAIC Festival in 2016. A portion of this interview appears below. Accompanying artwork appears on the following pages: 352, 359, and 377.

Q: *Can you tell us about the mediums you work in?*

CKP: I work in paper folding and paper engineering. Paper folding is the English way to say origami—folding without cutting, often from a square. In my own creative works, I use only a geometric style, as opposed to a figurative style. Then there's cutting, the paper engineering that comes from having laser cutters cut paper in different shapes that might fit together, thereby creating systems for making polyhedra without glue in a way that has some advantages over using glue. I also work with folding textiles, which is related. All the forms that I did in paper, I was able to transition the same patterns into cloth with a very different technique—a beautifully more accessible technique that does not require a high level of technical skill like it does in paper. I also work a lot in wood and plywood—what might be called flat-good engineering—where I work with laser cutters or CNC routers to develop systems for parts to be cut in 2-D but then assemble up from their special shapes. I like the game where the special shapes alone allow them be assembled without glue or connectors.

Q: *Which of those mediums do you enjoy most? Does one stand out for you?*

CKP: I like them all. Each has its own world of forms that are appropriate for it. I like to try to listen to the medium, finding the sweet spot of the medium and being in relationship with it. For instance, moving on from plywood to plastic and 3-D printing, and with a particular type of 3-D printing called fused deposition modeling where the plastic is coming out of the nozzle layer by layer. There are a lot of limits that govern how to use such lower-end 3-D printing machines that populate the marketplace. That's because each layer has to be in a relationship to the layer before or else it doesn't work so well. Without some other support structure, which in the low-end machine is the same material, one has to be a little bit careful how to use the printer. There is a sweet spot for which those machines are successful—thinking about that and designing for that is really stimulating to me.

Q: *You do some work with Middle Eastern patterns. Didn't you spend some influential time in Granada, Spain?*

CKP: Yes, I was there in 1990–91; it was incredibly influential. I arrived in the late summer and then went through the off-season, through the winter, and up to the late

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MSC: Primary 01A70; Amy L. Reimann (MR Author ID: 1118776) and David A. Reimann (MR Author ID: 912704)

spring/summer. A friend who was living there sent a postcard. She said, “I think you’d like it here.” I was a young, clueless, historically illiterate type of person, like young people tend to be. I showed up and didn’t know about Middle Eastern ornament. I had just the most glancing introduction right before I went. But not even on purpose, it was an accident. In a bookstore I picked up a book and saw a pattern and copied it and was like, “Wow, that was really amazing!” That was shortly before I went to Granada.

The Alhambra in Granada is one of the jewels of that civilization, and suddenly there I was, just immersed. I would go there and make drawings in a notebook. I didn’t have a camera and I think that was a real blessing. Without a camera it’s a different kind of attention. I learned how to take the fundamental region of the pattern, so that I could add more to it later. It was a self-disciplined very serious study where I would make drawings to learn this language. What was amazing is that I didn’t need a guide. I just gave attention to the objects and the compositions. It is incredibly beautiful that over those many centuries that kind of monument of information could speak and give itself to someone that gave attention to it. I could see all the decisions the craftsmen made from this kind of infinite possibility of composition space. Many separate guilds all working together. It was a civilizational high achievement.

Q: *Who are some of your other influences?*

CKP: I have a Japanese mentor, or a collection of them that I would say I have had relationship with. Both to be inspired by, mentored by, and work alongside in some ways. It’s sad but just recently the main figure in that world, Shuzo Fujimoto, sensei, passed away. He was considered a teacher. He had a long life. All of my folding work I would have to say is kind of like cultivating gardens that he begun. That’s the analogy that I like to use. He was extraordinarily playful and did explorations in many different kinds of folding styles. They’ve all influenced me and I’ve played in each. Like little new areas, he sort of started all the plots in the garden. Then I watered them and I might have added a plot here or there. Like jumping to cloth was kind of a significant thing, but I carried all of the same forms and things that had been heavily influenced by techniques and styles and little areas that he cultivated. It’s remarkable. I have been influenced by other people that were similarly inspired and they cultivated different kind of little subgenres of folding that he began. Tomoko Fuse and Toshikazu Kawasaki are a couple of the prominent ones that were also involved in introducing me to Shuzo Fujimoto and inviting me to Japan to meet him and to develop a relationship with him.

Q: *Let’s talk about the body of work you brought to MoSAIC.*

CKP: A lot of it is deployable and flat, and easy to pack and unroll. That’s really a nice thing. Like for instance this one here (Flower Tower, see Figure 1). It’s a kind of flower that comes from a nearly traditional thing, called a purse fold. Or in Japanese it’s called a tato. It’s nearly traditional because you can find similar things in leather coin purses in all kinds of cultures. It’s a really fascinating little spiral kind of structure. It’s actually a wrapper to wrap a polygon with a spiral structure. Another thing that’s kind of remarkable about it is that it’s the form that comes if you take eight pleats, that are kind of like lines and you shoot them into a vertex equally arranged. Then at that meeting of the pleats this little structure will happen, if you lay the pleats around in a spiral. You can’t really see that here because the edge of the paper is cut very short. But if you expanded this edge of the paper then you would see these pleats on a plane and then this little button in the center. I developed this technique for folding on top of this and then unfolding to create a new structure. The reason I got on this tangent is because it ships flat and then it comes out and displays in three dimensions. Depending

Figure 1 Flower Tower in paper, folded flat (left) and unfolded (center). Flower Tower in fabric, extended via thread (right).

on what I'm doing I might have a whole case then lay them out on a table and then they just deploy. That's a fun quality.

Q: *This is what we saw in the documentary* Between the Folds, *right?*

CKP: Yeah, this was the one, not this exact one, but I folded Flower Tower in that video. It's fun and it has the full flower tower structure that I play with. It's fun because it has some ability to open up and show that it's one piece, which is hidden when it's gathered together. You might not quite see that it's one piece.

For comparison, this one has 8 petals. This is also a flower tower. This was the first one. There's a whole space of composition just within this structure that has this quality where it's going to stand up and get recursively smaller and smaller.

Q: *Is this linen?*

CKP: This is silk. This is made in an entirely different method than the paper. But it has the same exact crease pattern. If you were to unfold it and mark all the mountains and the valleys, then you would see the same thing. What's remarkable about this, is that it doesn't really have the same structure as the paper. It's not sort of stiff when it stands up. It doesn't have the same kind of property. It just flops. It has the same shape but it doesn't do the same things. The cloth version is much more flexible.

Q: *What did you want to be when you grew up?*

CKP: I've always done art, so probably that was the strongest thread of my chaotic attention. I did origami a lot, and continued it past when you might do it as a child. I always was drawing. I did it in high school and was serious at it then. When I got to college I was kind of undefined and playing around and for a couple years didn't decide my major. But then when it came down to it, art was what meant the most to me and I said "Okay, let's do art and focus."

Editor's remark. Palmer's works, such as the Flower Tower, are interactive and have a surprising dynamic. You can see videos that demonstrate the dynamic nature of his work at the following links.

To see an 8-fold silk Flower Tower, click on `https://vimeo.com/18336512`.

A 12-fold paper Flower Tower appears at `https://vimeo.com/5172038`.

PROBLEMS

Proposals

To be considered for publication, solutions should be received by May 1, 2018.

2031. *Proposed by Barış Burçin Demir, Ankara, Turkey.*

Let n be an integer, $n \geq 2$. Let $A_1 A_2 A_3 \cdots A_{2n+1}$ be a regular polygon with $2n+1$ sides. Let P be the intersection of the segments $\overline{A_2 A_{n+2}}$ and $\overline{A_3 A_{n+3}}$. Prove that

$$(A_1 P)^2 = (A_2 A_3)^2 + (A_3 P)^2.$$

2032. *Proposed by Noah H. Rhee, University of Missouri–Kansas City, MO.*

Let a, b be real numbers with $a < b$, and let f be a continuous, strictly increasing function on the closed interval $[a, b]$. For $y \in \mathbb{R}$, define

$$E(y) = \int_a^b |f(x) - y| \, dx.$$

Prove that $E(y)$ has a minimum value as y varies in $\mathbb{R}$, and find all y for which the minimum is attained.

2033. *Proposed by Yoshihiro Tanaka, Hokkaido University, Sapporo, Japan.*

A deck is the collection of all 52 pairs (“cards”) of the form (n, s) where $1 \leq n \leq 13$ is the number on the card, and the suit s of the card is one of the symbols $\diamondsuit$, $\heartsuit$, $\spadesuit$, $\clubsuit$. Given an arbitrary partition of a deck into 13 sets $S_1, S_2, \ldots, S_{13}$ of 4 cards each, prove that there exists a corresponding partition C_1, C_2, C_3, C_4 of the deck into 4 sets of 13 cards each, such that each of the parts C_i $(1 \leq i \leq 4)$ satisfies:

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We invite readers to submit original problems appealing to students and teachers of advanced undergraduate mathematics. Proposals must always be accompanied by a solution and any relevant bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Proposals and solutions should be written in a style appropriate for this MAGAZINE.

Authors of proposals and solutions should send their contributions using the Magazine's submissions system hosted at `http://mathematicsmagazine.submittable.com`. *More detailed instructions are available there. We encourage submissions in PDF format, ideally accompanied by* LaTeX *source. General inquiries to the editors should be sent to* `mathmagproblems@maa.org`.

(i) C_i has one card from S_j for $1 \leq j \leq 13$, and
(ii) the cards in C_i all have different numbers.

2034. *Proposed by Julien Sorel, Piatra Neamt, PNI, Romania.*

Let $\mathfrak{C}$ be a circle. Two points A, B are independently chosen on the circumference of $\mathfrak{C}$, uniformly at random. Two further points C, D are independently chosen in the interior of $\mathfrak{C}$ uniformly at random. What is the probability that D shall lie inside $\triangle ABC$?

2035. *Proposed by Gregory Dresden, Prakriti Panthi (student), Anukriti Shrestha (student) and Jiahao Zhang (student), Washington & Lee University, Lexington, VA.*

Two real numbers x, y are said to *have a common decimal part* if $xy < 0$ and $x + y$ is an integer, or else $xy \geq 0$ and $x - y$ is an integer. More concretely, this means that the decimal expansions of x, y are of the forms

$$\pm a_m a_{m-1} \ldots a_1 a_0 \,.\, d_1 d_2 d_3 \ldots,$$
$$\pm b_n b_{n-1} \ldots b_1 b_0 \,.\, d_1 d_2 d_3 \ldots,$$

where the common decimal part is $0.d_1 d_2 d_3 \ldots$.

Find all polynomials of degree at least 2 with integer coefficients, all roots real, and irreducible over the rationals, whose roots have pairwise common decimal tails.

Quickies

1075. *Proposed by Raymond Mortini, Université de Lorraine and IECL, France.*

Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded and continuous function. Assume that there exist $a, b \in \mathbb{R}$ such that $f(a) \leq f(x) \leq f(b)$ for all $x \in \mathbb{R}$. Is it true that, for every $d > 0$, there exists a horizontal segment of length d with endpoints on the graph of f?

1076. *Proposed by Lokman Gökçe, Adana, Turkey.*

A quadrilateral $ABCD$ has angles $\angle ABC = 138°$, $\angle BAD = 108°$ and sides $AB = AD = \sqrt{3}\,BC$. What are the measures of angles $\angle ADC$ and $\angle BCD$?

Solutions

Consecutive successes in independent Bernoulli trials **October 2016**

2001. *Proposed by Herb Bailey and Dianne Evans, Rose-Hulman Institute of Technology, IN.*

Fix positive integers n and k. Numbers are drawn one at a time with replacement from an urn containing one of each of the first n positive integers. Find the expected number of drawings needed until k successive drawings are all ones.

Solution by Nicholas C. Singer, Annandale, VA.
Let E_k be the expected number of drawings. We claim that

$$E_k = \sum_{i=1}^{k} n^k = \begin{cases} k & (n = 1) \\ \frac{n^{k+1}-n}{n-1} & (n > 1). \end{cases} \tag{1}$$

Consider the following more general situation: Perform a sequence of independent Bernoulli trials ("coin tosses"), each with probability of success ("landing heads") equal to p ($0 \leq p \leq 1$). Thus, the problem is equivalent to finding the expected number E_k of independent Bernoulli trials with $p = 1/n$ that are needed to attain k consecutive successes, when a success is defined as drawing 1 from the urn.

We may as well consider the problem for all $k \geq 0$; in fact, $E_0 = 0$ since it takes no trials at all to ensure zero consecutive successes occur.

If $p = 1$ then every draw is a success, so $E_k = k$ and equation (1) holds for $n = 1$. Henceforth, assume $p < 1$.

For any $k \geq 0$, in order for $k + 1$ consecutive successes to occur, one must first obtain k consecutive successes, which requires an expected total of E_k trials. Given a chain of k consecutive successes, success on the next trial happens with probability p; under this condition, the expected number of trials needed to obtain $k + 1$ successive successes is equal to $E_k + 1$. With probability $1 - p$, on the other hand, a failure immediately follows a chain of k consecutive successes. In this case, it will still take E_{k+1} trials, in addition to the initial expected $E_k + 1$ trials that ended in a failure, to attain $k + 1$ consecutive successes; hence

$$E_{k+1} = p(E_k + 1) + (1 - p)(E_k + 1 + E_{k+1}).$$

Thus, we obtain the recursion $pE_{k+1} - E_k = 1$ for $k \geq 0$ with $E_0 = 0$, whose solution is

$$E_k = \frac{1 - p^k}{(1 - p)p^k}.$$

Taking $p = 1/n$, we see that equation (1) holds for $n > 1$ also.

Also solved by Robert A. Agnew, Michael Andreoli, Elton Bojaxhiu (Albania) & Enkel Hysnelaj (Australia), Robert Calcaterra, Robin Chapman (U.K.), John Christopher, Martin Getz & Yuanyuan Zhao, J. A. Grzesik, GWstat Problem Solving Group, Rituraj Nandan, Michael Nathanson, Northwestern University Math Problem Solving Group, Kostas Petrakos, Rob Pratt, Edward Schmeichel, Michael Vowe (Switzerland), Michael Woltermann, and the proposer. There was one incomplete or incorrect solution.

Dense-in-average real sequences **October 2016**

2002. *Proposed by Constantin P. Niculescu, Craiova, Romania and Gabriel T. Prăjitură, SUNY Brockport, NY.*

We will call a real sequence $\{x_n\}_{n\in\mathbb{N}_+}$ *dense in average* if

$$\left\{x_1, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2 + x_3}{3}, \ldots\right\}$$

is dense in $\mathbb{R}$.

(a) Show that there are sequences that are dense in $\mathbb{R}$ but not dense in average.

(b) Prove that every sequence that is dense in $\mathbb{R}$ has a subsequence that is both dense and dense in average.

Solution by Moubinool Omarjee, Lycée Henry IV, Paris, France.

(a) Let $\{q_n\}_{n\geq 1}$ be an enumeration of the countable set of positive rational numbers, and define the sequence $\{r_n\}_{n\geq 1}$ by $r_{2n-1} = q_n$, $r_{2n} = -q_n$ for $n \geq 1$. Clearly, $\{r_n\}$ is dense in $\mathbb{R}$. On the other hand, we have $\frac{1}{2n-1}\sum_{k=1}^{2n-1} r_k = q_n$ and $\frac{1}{2n}\sum_{k=1}^{2n} r_k = 0$, so the sequence of averages of $\{r_n\}$ has no negative terms; therefore, $\{r_n\}$ is not dense in average.

(b) We prove that any sequence $\{x_n\}_{n\geq 1}$ dense in $\mathbb{R}$ has a subsequence $\{x_{n_i}\}_{i\geq 1}$ that is both dense and dense in average. Let $\{I_j\}_{j\geq 1}$ be an enumeration of all nonempty open intervals with rational endpoints. Each I_j is of the form (r_j, s_j) with rational numbers $r_j < s_j$. By density of $\{x_n\}$, we may choose n_1 with $x_{n_1} \in I_1$. Solutions y to the inequality $r_1 < (x_{n_1} + y)/2 < s_1$ lie in a nonempty open interval so, by density of $\{x_n\}_{n>n_1}$, there exists $n_2 > n_1$ such that $(x_{n_1} + x_{n_2})/2 \in I_1$. Similarly, once $n_1 < n_2 < \cdots < n_{2j-1} < n_{2j}$ are chosen, by density of $\{x_n\}_{n>n_{2j}}$, we can choose n_{2j+1} and n_{2j+2} with $n_{2j} < n_{2j+1} < n_{2j+2}$ such that $x_{n_{2j+1}} \in I_{j+1}$ and $(x_{n_1} + x_{n_2} + \cdots + x_{n_{2j+1}} + x_{n_{2j+2}})/(2j+2) \in I_{j+1}$. Clearly, the sequence $\{x_{n_j}\}_{j\geq 1}$ is both dense and dense in average.

Also solved by Elton Bojaxhiu (Albania) & Enkel Hysnelaj (Australia), Robert Calcaterra, Robin Chapman (UK), C. J. Dowd (Student), Eugene A. Herman, Jody Lockhart, Northwestern University Math Problem Solving Group, Celia Schacht (Student), Edward Schmeichel, and the proposer. There was one incomplete or incorrect solution.

The tricky derivative of a trigonometric integral **October 2016**

2003. *Proposed by Julien Sorel, Columbus, GA.*

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(0) = 0$, and

$$f(x) = \int_0^x \cos\frac{1}{t}\,\cos\frac{3}{t}\,\cos\frac{5}{t}\,\cos\frac{7}{t}\,dt$$

for $x \neq 0$. Show that f is differentiable and $f'(0) = 1/8$.

Solution by Northwestern University Math Problem Solving Group, Evanston, IL.

Lemma. If $\alpha \neq 0$, then $\lim_{x\to 0} \frac{1}{x}\int_0^x e^{i\alpha/t}\,dt = 0$.

Proof. First we prove $\lim_{x\to 0^+} \frac{1}{x}\int_0^x e^{i\alpha/t}\,dt = 0$ (limit from the right). Assume $x > 0$. We have $xe^{i\alpha/x} \to 0$, and also

$$\left|\frac{1}{x}\int_{\frac{1}{x}}^{\infty} e^{i\alpha u}\frac{du}{u^3}\right| \leq \frac{1}{x}\int_{\frac{1}{x}}^{\infty}\frac{du}{u^3} = \frac{x}{2} \to 0 \quad \text{as } x \to 0^+.$$

Performing the change of variable $t = 1/u$ and integrating by parts:

$$\frac{1}{x}\int_0^x e^{i\alpha/t}\,dt = \frac{1}{x}\int_{\frac{1}{x}}^{\infty} e^{i\alpha u}\frac{du}{u^2} = \frac{i}{\alpha}xe^{i\alpha/x} + \frac{2}{i\alpha x}\int_{\frac{1}{x}}^{\infty} e^{i\alpha u}\frac{du}{u^3} \to 0 \quad \text{as } x \to 0^+.$$

For the limit from the left, we have $\lim_{x\to 0^-} \frac{1}{x}\int_0^x e^{i\alpha/t}\,dt = \lim_{x\to 0^+} \frac{1}{x}\int_0^x e^{-i\alpha/t}\,dt = 0$, so the lemma follows.

Now we prove the assertions in the statement of the problem. By the fundamental theorem of calculus, $f(x)$ is differentiable for every $x \neq 0$, so we need only prove that f is differentiable at $x = 0$ and compute $f'(0)$. Since $\cos u = \frac{1}{2}(e^{iu} + e^{-iu})$, we have

$$\cos\frac{1}{t}\cos\frac{3}{t}\cos\frac{5}{t}\cos\frac{7}{t} = \frac{e^{i/t} + e^{-i/t}}{2}\cdot\frac{e^{3i/t} + e^{-3i/t}}{2}\cdot\frac{e^{5i/t} + e^{-5i/t}}{2}\cdot\frac{e^{7i/t} + e^{-7i/t}}{2}$$

$$= \frac{1}{16}\sum_{\varepsilon_i\in\{+1,-1\}} e^{i(\varepsilon_0+3\varepsilon_1+5\varepsilon_2+7\varepsilon_3)/t} = \frac{1}{16}(2 + g(t)).$$

Except for the two constant terms $\exp[(i - 3i - 5i + 7i)/t] = \exp(0) = 1 = \exp[(-i + 3i + 5i - 7i)/t]$ in the sum above, the remaining terms are of the form $e^{i\alpha/t}$ with $\alpha \neq 0$

and their sum is $g(t)$. Using the lemma above, we have $\lim_{x\to 0} \frac{1}{x}\int_0^x g(t)\,dt = 0$. Since $f(0) = 0$, we have

$$f'(0) = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{1}{x}\int_0^x \frac{1}{8}\,dx + \frac{1}{16}\lim_{x\to 0}\frac{1}{x}\int_0^x g(t)\,dt$$
$$= \lim_{x\to 0} \frac{1}{x}\cdot\frac{x}{8} + \frac{1}{16}\cdot 0 = \frac{1}{8}.$$

Also solved by Michel Bataille (France), M. Bello & M. Benito & O. Ciaurri & E. Fernández & L. Roncal (Spain), Robin Chapman (UK), Robert Calcaterra, Bruce E. Davis, Robert L. Doucette, Eugene A. Herman, Lee-Wai Lau (Hong Kong), Moubinool Omarjee (France), Paolo Perfetti (Italy), Rudolf Rupp (Germany), Edward Schmeichel, and the proposer. There were 2 incomplete or incorrect solutions.

A binomial identity **October 2016**

2004. *Proposed by Mihàly Bencze, Brasov, Romania.*

For every positive integer n, prove the identity:

$$\sum_{k=1}^{n-1}\binom{n}{k}\frac{kn^{n-k}}{k+1} = \frac{n(n^n-1)}{n+1}.$$

Solution by Robert L. Doucette, McNeese State University, Lake Charles, LA.
Using the well-known identity

$$\frac{1}{k+1}\binom{n}{k} = \frac{1}{n+1}\binom{n+1}{k+1} \quad (n, k \geq 0)$$

plus the binomial theorem, we have

$$\sum_{k=1}^{n-1}\binom{n}{k}\frac{kn^{n-k}}{k+1} = \sum_{k=0}^{n}\binom{n}{k}\frac{kn^{n-k}}{k+1} - \frac{n}{n+1}$$
$$= \sum_{k=0}^{n}\binom{n}{k}n^{n-k} - \sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}n^{n-k} - \frac{n}{n+1}$$
$$= (n+1)^n - \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}n^{(n+1)-(k+1)} - \frac{n}{n+1}$$
$$= (n+1)^n - \frac{(n+1)^{n+1}-n^{n+1}}{n+1} - \frac{n}{n+1}$$
$$= \frac{n(n^n-1)}{n+1}.$$

Also solved by Robert A. Agnew, Adnan Ali (India), Michel Bataille (France), Elton Bojaxhiu (Albania) & Enkel Hysnelaj (Australia), Khristo N. Boyadzhiev, Brian Bradie, Robert Calcaterra, Robin Chapman (UK), Travis D. Cunningham, Bruce E. Davis, Prithwijit De (India), Ross Dempsey, Saumya Dubey, Nikolas Dunn, Habib Y. Far, Dmitry Fleischman, Michael Goldenberg & Mark Kaplan, Raymond N. Greenwell, J.A. Grzesik, GWstat Problem Solving Group, Kyle Hansen, Eugene A. Herman, Junah Kang & Junhee Park (South Korea), Harris Kwong, Tsz Kit Lau, Christopher Mbakwe & Isaac Wass, Jerry Minkus, Rituraj Nandan, Michael Nathanson, Northwestern University Math Problem Solving Group, Moubinool Omarjee (France), Ángel Plaza (Spain), Henry Ricardo, Adnan H. Sabuwala, Edward Schmeichel, Ahmad Talafha & Kevin Wunderlich, Timothy Woodcock, John Zacharias, Yuanyuan Zhao, and the proposer.

A series of reciprocals of products of integers **October 2016**

2005. *Proposed by Daniel Ullman, George Washington University, Washington, DC.*

Let $S = \{3 \cdot 2^k - 2 : k \in \mathbb{N}\} = \{1, 4, 10, \dots\}$. For every nonempty finite subset $A \subset \mathbb{N}$, let $\pi(A) = \prod_{k \in A} k$. Compute

$$\sum \frac{1}{\pi(A)}$$

where the sum is taken over all finite nonempty subsets A of S.

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY.
The value of the sum is 2. We have

$$T := \sum_{\emptyset \neq A \subseteq S} \frac{1}{\pi(A)} = \sum_{\ell=1}^{\infty} \sum_{\substack{k_i \in S \\ k_i \text{ distinct}}} \frac{1}{k_1 k_2 \cdots k_\ell}.$$

Hence

$$T = -1 + \prod_{k \in S}\left(1 + \frac{1}{k}\right) = -1 + \prod_{i=0}^{\infty}\left(1 + \frac{1}{3 \cdot 2^i - 2}\right) = -1 + \lim_{n \to \infty} t_n,$$

where t_n is the telescoping product

$$t_n := \prod_{i=0}^{n} \frac{3 \cdot 2^i - 1}{3 \cdot 2^i - 2} = \prod_{i=0}^{n} \frac{3 \cdot 2^i - 1}{2(3 \cdot 2^{i-1} - 1)} = \frac{3 \cdot 2^n - 1}{2^{n+1}(3 \cdot 2^{-1} - 1)} = 3 - 2^{-n}.$$

Hence,

$$\sum_{\emptyset \neq A \subseteq S} \frac{1}{\pi(A)} = -1 + \lim_{n \to \infty} t_n = -1 + 3 = 2.$$

Also solved by Adnan Ali, Mildred Asamoah, Elton Bojaxhiu (Albania) & Enkel Hysnelaj (Australia), Christopher J. Dowd, Robert Doucette, Eugene Herman, Peter McPolin, Rituraj Nandan, Northwestern University Math Problem Solving Group, Rob Pratt, Edward Schmeichel, Nicholas Singer, John Zacharias, and the proposer. There were 3 incomplete or incorrect solutions.

An optimization problem in a regular n-gon **December 2016**

2006. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu", Bîrlad, Romania.*

Let $\mathcal{F}$ be a regular polygon whose n vertices $A_1, A_2, \dots, A_n$ lie on the unit circle. For any point P on the plane of $\mathcal{F}$, let $f(P) = PA_1 \cdot PA_2 \cdots PA_n$ be the product of the distances from P to the vertices of $\mathcal{F}$. Find the maximum value of $f(P)$ over all points P lying in the interior of, or on any of the sides of $\mathcal{F}$. For which position(s) of P is this maximum attained?

Solution by Marty Getz, Dixon Jones and Yuanyuan Zhao, University of Alaska Fairbanks, AK.
We show that the maximum of f is equal to $1 + \cos^n(\pi/n)$ and it is attained precisely when P is a midpoint of a side of $\mathcal{F}$.

Without loss of generality, let $\mathcal{F}$ be the polygon on the complex plane whose kth vertex A_k lies at $z_k = \exp(i(2k-1)\pi/n)$ for $k = 1, 2, \dots, n$. Let z be the complex coordinate of an arbitrary point P on the complex plane. The roots of the polynomial $z^n + 1$ are precisely $z_1, \dots, z_n$, hence

$$f(P) = \prod_{k=1}^{n} PA_k = \prod_{k=1}^{n} |z - z_k| = |z^n + 1|.$$

Let $g(z) = (f(P))^2 = |z^n + 1|^2 = |z|^{2n} - 2\Re(z^n) + 1$. We shall find the maximum of g in the interior of, or on any of the sides of $\mathcal{F}$. Let x, θ be the real part and the argument of z, respectively, so $z = xe^{i\theta}\sec\theta$. By symmetry, it suffices to consider the triangular sector $\mathcal{T}$ given by $0 \le x \le \cos\frac{\pi}{n}$, $0 \le \theta \le \frac{\pi}{n}$. Note that the continuous function g must attain its maximum on $\mathcal{T}$ since $\mathcal{T}$ is closed and bounded, hence compact. Writing g as a function of x and θ, we have

$$g(x, \theta) = x^{2n}\sec^{2n}\theta + 2x^n \sec^n\theta \cos n\theta + 1.$$

Differentiating with respect to θ:

$$\begin{aligned} g_\theta &= 2nx^{2n}\sec^{2n}\theta\tan\theta + 2nx^n\sec^n\theta\tan\theta\cos n\theta - 2nx^n\sec^n\theta\sin n\theta \\ &= 2nx^n[x^n\sin\theta + \cos^n\theta\sin\theta\cos n\theta - \cos^n\theta\cos\theta\sin n\theta]\sec^{2n+1}\theta \\ &= 2nx^n\big[x^n\sin\theta - \sin\big((n-1)\theta\big)\cos^n\theta\big]\sec^{2n+1}\theta. \end{aligned}$$

Since $0 \le x \le \cos\frac{\pi}{n}$ and $0 \le \theta \le \frac{\pi}{n}$, we have $\theta \le (n-1)\theta \le \pi - \theta$, hence $\sin\theta \le \sin\big((n-1)\theta\big)$ and $x^n \le \cos^n\frac{\pi}{n} \le \cos^n\theta$. Thus, we have $g_\theta \le 0$; moreover, $g_\theta = 0$ precisely when $x = 0$ or $\theta = 0$. Hence, the maximum of g on $\mathcal{T}$ must be attained on the horizontal side $\theta = 0$, or else on the "side" $x = 0$—which degenerates to the single point $z = 0$. Thus, it remains to find the maximum value of $g(x, 0) = x^{2n} + 2x^n + 1 = (1 + x^n)^2$ with $x \in [0, \cos\frac{\pi}{n}]$, which is obviously attained when $x = \cos\frac{\pi}{n}$ since $g(x, 0)$ is strictly increasing for $x \ge 0$. Geometrically, the maximum value of g, and hence of f, on the triangle $\mathcal{T}$, is attained at the vertex $x = \cos(\pi/n)$, $\theta = 0$, which is the midpoint of the side P_1P_n. By symmetry, the maximum of f is attained precisely at the midpoint

$$\sqrt{g\left(\cos\frac{\pi}{n}, 0\right)} = 1 + \cos^n\frac{\pi}{n}.$$

Also solved by Armstrong Problem Solvers, Elton Bojaxhiu (Albania) & Enkel Hysnelaj (Australia), Dmitry Gokhman, and the proposer. There was one incomplete or incorrect solution.

A limit of integrals with value $(f(0) + f(1))/2$ December 2016

2007. *Proposed by Ángel Plaza, Department of Mathematics, University Las Palmas de Gran Canaria, Spain.*

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Evaluate

$$\lim_{n\to\infty}\left(n \cdot \int_0^1 \left(\frac{2\left(x - \frac{1}{2}\right)^2}{x^2 - x + \frac{1}{2}}\right)^n f(x)\,\mathrm{d}x\right).$$

Solution by Hongwei Chen, Christopher Newport University, Newport News, VA.

The limit is equal to $(f(0) + f(1))/2$. First, we prove an auxiliary result.

Lemma. Let $a > 0$ and let $G(x)$ be a nonnegative decreasing function on $[0, a]$, differentiable at $x = 0$, with $G(0) = 1$ and $G'(0) < 0$. If $F(x)$ is continuous on $[0, a]$, then

$$\lim_{n\to\infty}\int_0^a nG^n(x)F(x)\,dx = -\frac{F(0)}{G'(0)}.$$

We prove the lemma. Let $\varphi = F(0)$ and $\gamma = -G'(0) > 0$. Let $\varepsilon \in (0, \gamma)$ be arbitrary. Since $F(x)$ is continuous at $x = 0$, there is $\delta_1 \in (0, a)$ such that

$$\varphi - \varepsilon \le F(x) \le \varphi + \varepsilon \quad \text{for all } x \in [0, \delta_1].$$

Similarly, since $G(0) = 1$ and $G'(0) = -\gamma$, there exists $\delta_2 \in (0, a)$ such that

$$(1 - \gamma x) - \varepsilon x \le G(x) \le (1 - \gamma x) + \varepsilon x \quad \text{for all } x \in [0, \delta_2].$$

Let $\delta = \min\{\delta_1, \delta_2, 1/\gamma, 1/(\gamma + \varepsilon)\}$. Note that δ depends only on F, G, and ε. Let L be the value of the limit in the lemma. Clearly, $L = L_1 + L_2$ where L_1, L_2 are the limits of the integral over $[0, \delta]$ and over $[\delta, a]$, respectively, which we compute separately.

We have $0 \le G(\delta) < G(0) = 1$ inasmuch as G is nonnegative and decreasing with $G'(0) < 0$. Since the continuous function F is bounded on $[0, a]$, it follows that

$$\left| \int_\delta^a nG^n(x)F(x)\,dx \right| \le an\,|G(\delta)|^n \cdot \max_{x \in [0,a]} |F(x)| \to 0 \quad \text{as } n \to \infty,$$

hence $L_2 = 0$. On the other hand, the assumptions $0 < \varepsilon < \gamma$ and $\delta \le 1/(\gamma + \varepsilon)$ ensure that $1 - (\gamma \pm \varepsilon)\delta \in [0, 1)$, so

$$b_n(-\varepsilon) \le G^n(x)F(x) \le b_n(\varepsilon) \quad \text{for } x \in [0, \delta],$$

where $b_n(\varepsilon) := [1 - (\gamma - \varepsilon)x]^n(\varphi + \varepsilon)$. Multiplying by n and integrating over $[0, \delta]$, we obtain

$$B_n(-\varepsilon) \le \int_0^\delta nG^n(x)F(x)dx \le B_n(\varepsilon),$$

where

$$B_n(\varepsilon) := \int_0^\delta nb(\varepsilon)dx = \frac{n(\varphi + \varepsilon)\{1 - [1 - (\gamma - \varepsilon)\delta]^{n+1}\}}{(n+1)(\gamma - \varepsilon)}.$$

Clearly, $B_n(\varepsilon) \to (\varphi + \varepsilon)/(\gamma - \varepsilon)$ as $n \to \infty$, so

$$\frac{\varphi - \varepsilon}{\gamma + \varepsilon} \le \liminf_{n\to\infty} \int_0^\delta nG^n(x)F(x)dx \quad \text{and} \quad \limsup_{n\to\infty} \int_0^\delta nG^n(x)F(x)dx \le \frac{\varphi + \varepsilon}{\gamma - \varepsilon}.$$

Letting $\varepsilon \to 0$ we see that $L_1 = \varphi/\gamma$, hence $L = L_1 + L_2 = L_1 = \varphi/\gamma = -F(0)/G'(0)$, concluding the proof of the lemma.

To solve the problem, we compute the limits l_1, l_2, respectively, of n times the integrals over $[0, 1/2]$ and $[1/2, 0]$. First, take $F(x) = f(x)$ and $G(x) = 2(x - 1/2)^2/(x^2 - x + 1/2)$ on $[0, 1/2]$. The conditions of the lemma are satisfied and we have $G'(0) = -2$, so $l_1 = F(0)/2 = f(0)/2$. Next, take $F(x) = f(1 - x)$ and the same $G(x)$ $(= G(1 - x))$ on $[0, 1/2]$ to get $l_2 = F(0)/2 = f(1)/2$. We conclude that $l = l_1 + l_2 = (f(0) + f(1))/2$.

Also solved by Ulrich Abel (Germany), Michel Bataille (France), Lixing Han, Eugene A. Herman, Rituraj Nandan, Northwestern University Math Problem Solving Group, Paolo Perfetti (Italy), and the proposer. There were 3 incomplete or incorrect solutions.

An irrational inequality **December 2016**

2008. *Proposed by Michael W. Botsko, Saint Vincent College, Latrobe, PA.*

Is there a function $f : \mathbb{R} \to (0, \infty)$ such that the inequality

$$\frac{f(x)}{f(y)} \le |x - y|$$

holds for all real numbers x, y such that x is irrational and y rational?

Solution by Paul Budney, Sunderland, MA.
We show that no such function exists. Consider such a hypothetical function f. For y rational and n a positive integer, let $B_n(y) = \big(y, y + 1/(nf(y))\big)$ (an open interval in $\mathbb{R}$). For irrational $x \in B_n(y)$ we have $f(x) \le f(y)(x - y) < f(y)/(nf(y)) = 1/n$ by the assumption on f. Let $\{y_k\}$ be an enumeration of the rationals. The set $C_n = \bigcup_k B_n(y_k)$ is clearly open. In fact, C_n is dense in $\mathbb{R}$ because any nonempty open interval I contains some rational number y, and hence $I \cap B_n(y)$ is a nonempty subset of C_n. Each set $C_n - \{y_n\}$ is still open and dense in $\mathbb{R}$. By the Baire category theorem, the set $D = \bigcap_n (C_n - \{y_n\})$ is dense in $\mathbb{R}$ Let $x \in D$. Since D contains no element in $\{y_n\} = \mathbb{Q}$ by construction, it follows that x is irrational. For each n we have $x \in C_n$, hence $x \in B_n(y)$ for some $y \in \mathbb{Q}$ and therefore $f(x) < 1/n$. This contradicts the hypothesis $f(x) > 0$, showing that such a function f does not exist.

Also solved by Northwestern University Math Problem Solving Group, Souvik Dey (India), Edward Schmeichel, and the proposer. There were 5 incomplete or incorrect solutions.

Rings of functions à la von Neumann **December 2016**

2009. *Proposed by Greg Oman, University of Colorado, Colorado Springs, CO.*

Consider any metric space (X, d). Let $\mathcal{F}_X$ be the collection of all functions $f : X \to \mathbb{R}$ (not necessarily continuous). The set $\mathcal{F}_X$ possesses natural operations of addition and multiplication, namely the sum $f + g$ and product fg of two elements f, g of $\mathcal{F}_X$ are characterized by the identities

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (fg)(x) = f(x)g(x), \quad \text{for all } x \in X.$$

Endowed with these operations, $\mathcal{F}_X$ is a ring. Since the sum and product of continuous real functions are continuous, the set $\mathcal{C}_X$ consisting of all continuous functions in $\mathcal{F}_X$ is a subring of $\mathcal{F}_X$. Is there a metric space (X, d) such that $\mathcal{C}_X$ isomorphic to $\mathcal{F}_X$ as rings, but $\mathcal{C}_X$ is a proper subset of $\mathcal{F}_X$?

Solution by Missouri State University Problem Solving Group, Springfield, MO.
We prove that no such metric space (X, d) exists. The ring $\mathcal{F}_X$ is a von Neumann regular ring, namely for every $f \in \mathcal{F}_X$ there exists $g \in \mathcal{F}_X$ such that $fgf = f$. Indeed, given f, it suffices to define g by

$$g(x) = \begin{cases} 1/f(x) & \text{if } f(x) \neq 0, \\ 0 & \text{if } f(x) = 0. \end{cases}$$

Assuming that $\mathcal{C}_X$ is isomorphic to $\mathcal{F}_X$, it must also be a von Neumann regular ring. Let $p \in X$ be arbitrary. The function $f(x) = d(x, p)$ is continuous, hence $fgf = f$ for some $g \in \mathcal{C}_X$, so the identity $d(x, p)g(x) = 1$ holds for $x \neq p$. Since g is continuous at p by hypothesis, the point p must be isolated in X; otherwise, $\lim_{x \to p} g(x) = g(p)$ and $\lim_{x \to p} d(x, p) = 0$ would both exist, and the identity above gives $0 \cdot g(p) = 1$, which is absurd. We conclude that the isomorphism $\mathcal{F}_X \cong \mathcal{C}_X$ implies that every point of X is isolated, hence X is a discrete metric space and every function on X is continuous, so $\mathcal{C}_X = \mathcal{F}_X$ in this case.

Also solved by Souvik Dey (India), Northwestern University Math Problem Solving Group, and the proposer.

The number of cycles of a random permutation **December 2016**

2010. *Proposed by Mehtaab Sawhney (student), University of Pennsylvania, Philadelphia, PA.*

Let k be a positive integer. Consider the experiment of choosing a permutation π of k objects uniformly at random (i.e., any two permutations σ, π are equally likely to be chosen). Let N be the number of cycles of π. Find the expected value $\mathbf{E}[N2^N]$ of the random variable $N2^N$, as a function of k.

Solution by Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Australia).
We shall take k to be a nonnegative integer. We use the following notations: $\mathbf{P}_k\big[\mathcal{E}\big]$ is the probability of an event $\mathcal{E} \subset \mathfrak{S}_k$ (the set of permutations of the numbers $1, 2, \ldots, k$, regarded as a uniform probability space), while $\mathbf{E}_k[X]$ is the expected value of a random variable X on $\mathfrak{S}_k$. The random variable N is the number of cycles of a random permutation π, while Λ is the length of the cycle of π which contains the element 1. Positive integers not exceeding k are possible values of N and Λ. (When $k = 0$, the space $\mathfrak{S}_k$ consists of only the empty permutation $\pi = ()$, so $N = 0$. On the other hand, Λ is undefined when $k = 0$.) We prove that

$$f_k := \mathbf{E}_k\big[N2^N\big] = 2(k+1)(H_{k+1} - 1),$$

where H_n is the harmonic sum $H_n = 1 + 1/2 + \cdots + 1/n$.

For $1 \le \lambda \le k$, we have

$$\mathbf{P}_k[\Lambda = \lambda] = \frac{\binom{k-1}{\lambda-1} \cdot (\lambda-1)! \cdot (k-\lambda)!}{k!} = \frac{1}{k},$$

since the $\lambda - 1$ elements of the cycle including 1 (other than 1 itself) may be chosen in $\binom{k-1}{\lambda-1}$ ways, then there are $(\lambda-1)!$ possible cycles on these elements plus 1, and $(k-\lambda)!$ ways to permute the remaining elements.

We compute $g_k = \mathbf{E}_k\big[2^N\big]$. Clearly, $g_0 = \mathbf{E}_0\big[2^0\big] = 1$, while for $k \ge 1$:

$$g_k = \sum_{\lambda=1}^{k} \mathbf{P}_k[\Lambda = \lambda]\, \mathbf{E}_k\big[2^N \mid \Lambda = \lambda\big] = \sum_{\lambda=1}^{k} \frac{1}{k} \cdot \mathbf{E}_k\big[2^N \mid \Lambda = \lambda\big].$$

Upon removing the λ-cycle including 1, a uniform random permutation of the remaining $k - \lambda$ elements is obtained, having $N_{k-\lambda} = N_k - 1$ cycles, so we have

$$g_k = \frac{1}{k}\sum_{\lambda=1}^{k} \mathbf{E}_{k-\lambda}\big[2^{1+N}\big] = \frac{1}{k}\sum_{\lambda=1}^{k} \mathbf{E}_{k-\lambda}\big[2 \cdot 2^N\big] = \frac{2}{k}\sum_{\lambda=1}^{n} g_{k-\lambda} \quad \text{for } k \ge 1.$$

Since $g_0 = 1$, it follows from this recurrence and induction that $g_k = k + 1$ for $k \ge 0$.

Next, let $f_k = \mathbf{E}_k\big[N2^N\big]$ be the expected value we seek to compute. Clearly, $f_0 = \mathbf{E}_0\big[0 \cdot 2^0\big] = 0$ and $f_1 = \mathbf{E}_1\big[1 \cdot 2^1\big] = 2$. For $k \ge 2$:

$$f_k = \sum_{\lambda=1}^{k} \mathbf{P}_k[\Lambda = \lambda]\, \mathbf{E}_k\big[N2^N \mid \Lambda = \lambda\big] = \sum_{\lambda=1}^{k} \frac{1}{k} \cdot \mathbf{E}_{k-\lambda}\big[(1+N)2^{1+N}\big]$$

(by the argument in the evaluation of g_k above)

$$= \frac{2}{k}\sum_{\lambda=1}^{k}\left(\mathbf{E}_{k-\lambda}\left[2^N\right]+\mathbf{E}_{k-\lambda}\left[N2^N\right]\right) = \frac{2}{k}\sum_{\lambda=1}^{k}[g_{k-\lambda}+f_{k-\lambda}]$$

$$= \frac{2}{k}\sum_{\lambda=1}^{k}[(k-\lambda+1)+f(k-\lambda)] \quad \text{(by the evaluation of } g_k \text{ above)}$$

$$= \frac{2}{k}\cdot\frac{k(k+1)}{2}+\frac{2}{k}\sum_{\lambda=1}^{k} f_{k-\lambda} = (k+1)+\frac{2}{k}\sum_{\lambda=1}^{k} f_{k-\lambda}.$$

Thus, we have

$$kf_k = k(k+1)+2\sum_{l=0}^{k-1} f_l$$

for $k \geq 1$. (The case $k=1$ does hold since $f_0 = 0$ and $f_1 = 2$.) Hence, for $k \geq 2$,

$$kf_k-(k-1)f_{k-1} = [k(k+1)-k(k-1)]+2f_{k-1} = 2f_{k-1}+2k,$$

whence we have the recurrence formula:

$$f_k = 2+\frac{k+1}{k}f_{k-1} \quad \text{for } k \geq 1.$$

(The case $k=1$ holds by direct verification.) From the recurrence above and the values $f_0 = 0$, $f_1 = 2$, the formula

$$f_k = 2(k+1)(H_{k+1}-1)$$

follows by routine induction.

Editor's Note. Most submissions used the well-known "rising factorial" generating function $x^{\overline{k}} = x(x+1)(x+2)\cdots(x+k-1) = \sum_{N=0}^{k}\left[{k \atop N}\right]x^N$ for the number $\left[{k \atop N}\right]$ of permutations of k objects having exactly N cycles. The solution above, although less expeditious, offers a more direct combinatorial argument.

Also solved by Robert A. Agnew, Robin Chapman, Isaac Garfinkle, GWstat Problem Solving Group, José H. Nieto (Venezuela), Northwestern University Math Problem Solving Group, Rob Pratt, Edward Schmeichel, and the proposer.

Answers

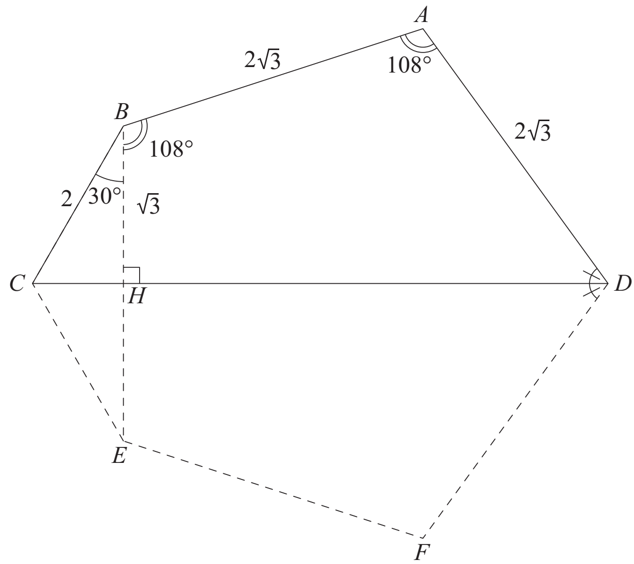
Solutions to the Quickies from page 384.

A1075. The answer is yes. Define $H(x) := f(x)-f(x+d)$. We have $H(a) = f(a)-f(a+d) \leq 0$ and $H(b) = f(b)-f(b+d) \geq 0$. By the intermediate value theorem, there exists $x_0 \in [a,b]$ such that $H(x_0) = 0$, hence $y_0 := f(x_0) = f(x_0+d)$, so the segment L from (x_0, y_0) to (x_0+d, y_0) is as required.

Editor's Note. The assertion obviously fails for any strictly increasing function f. Thus, it is not enough to assume that f is bounded, as the function $f(x) = \arctan x$ shows. This Quickie was inspired by Peter Horak, Partitioning $\mathbb{R}^n$ into Connected Components, *Amer. Math. Monthly* **122** no. 3 (2015) 280–283.

A1076. Draw the quadrilateral $ABCD$ as well as the regular pentagon $BADFE$. Without loss of generality take $BC = 2$, so $AB = 2\sqrt{3}$ and we have $\angle CBE = \angle ABC -$

$\angle ABE = 138^\circ - 108^\circ = 30^\circ$. Let H be the midpoint of $\overline{BE}$ so $\overleftrightarrow{DH}$ is an axis of symmetry of the pentagon $BADFE$ and $BH \perp DH$. Since $BH/BC = \sqrt{3}/2 = \sin 30^\circ$, it follows that C lies on $\overleftrightarrow{DH}$, so in right triangle $\triangle BHC$ we have $\angle BCD = \angle BCH = 90^\circ - 30^\circ = 60^\circ$. Finally, $\overline{DC}$ bisects angle $\angle ADF$, so $\angle ADC = 108^\circ/2 = 54^\circ$.



SOLUTION TO PINEMI PUZZLE FROM PAGE 337.

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REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles, books, and other materials are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Smith, Deborah, Mathematical mystery of ancient clay tablet solved, https://newsroom.unsw.edu.au/news/science-tech/mathematical-mystery-ancient-clay-tablet-solved.

Mansfield, Daniel F., and N. J. Wildberger, Plimpton 322 is Babylonian exact sexagesimal trigonometry, *Historia Mathematica* 44 (4) (November 2017) 395–419, http://dex.doi.org/10.1016/j.hm.2017.08.001.

Mansfield, Daniel F., Ancient Babylonian tablet—world's first trig table, video (2:08), https://www.youtube.com/watch?v=i9-ZPGp1AJE.

Lamb, Evelyn, Don't fall for Babylonian trigonometry hype: Separating fact from speculation in math history, https://blogs.scientificamerican.com/roots-of-unity/dont-fall-for-babylonian-trigonometry-hype/.

Plimpton 322 is a small Babylonian clay tablet almost 4,000 years old that contains 15 rows of Pythagorean triples, corresponding to increasingly flatter right triangles. The tablet and its contents has been known for a century; but what was its purpose, and why those particular triangles, and how were the contents calculated? Despite the fact that measurement of angles in degrees originated with the Babylonians, the tablet contains no reference to angles. Authors Wildberger and Mansfield refer to the tablet as a "the world's oldest trigonometric table... of a completely unfamiliar kind... ahead of its time by thousands of years." Wildberger claims that it illustrates "a simpler, more accurate trigonometry that has clear advantages over our own," advantages that he espoused in his *Divine Proportions: Rational Trigonometry to Universal Geometry* (2005). And Mansfield speaks of "enormous potential for applications to surveying, computers, and education." That is most unlikely. What we have is a very old artifact hyped by a slick new public relations team.

Byrne, Oliver, *Euclid's Elements of Geometry: Completing Oliver Byrne's Work*, Kronecker Wallis (Barcelona), to be published December 2018, https://www.kickstarter.com/projects/1174653512/euclids-elements-completing-oliver-byrnes-work.

Swetz, Frank J., and Victor J. Katz, Mathematical Treasures—Oliver Byrne's Euclid, https://www.maa.org/press/periodicals/convergence/mathematical-treasures-oliver-byrnes-euclid.

Oliver Byrne's edition of Euclid, https://www.math.ubc.ca/~cass/Euclid/byrne.html.

Oliver Byrne is famous for rewriting the first six books of Euclid using colors and illustrations in place of symbols, and labels for lengths, angles, and areas. Several facsimiles of his 1847 book have been published in recent years, and the work is available online (thanks to the University of British Columbia). The publisher Kronecker Wallis in Barcelona has proposed extending his work to a similar treatment of the remaining seven books of Euclid and publishing the completed work in 2018. Notable is that funding for the project was obtained by crowd-sourcing through Kickstarter, with the $138K goal more than doubled by 1,400 backers.

Math. Mag. **90** (2017) 395–396. doi:10.4169/math.mag.90.5.395.

Binkley, Collin, Math experts join brainpower to help address gerrymandering, https://www.usnews.com/news/best-states/wisconsin/articles/2017-08-18/math-experts-join-brainpower-to-help-address-gerrymandering.

Metric Geometry and Gerrymandering Group (MGGG), https://sites.tufts.edu/gerrymandr/.

Mander, Gerry [sic], Geometry of Redistricting Workshop August 7-9th 2017, https://www.youtube.com/playlist?list=PLr7G5jnVFYLiTpEiQkQB_FyQ372oSO8Au.

Klarreich, Erica, How to quantify (and fight) gerrymandering, https://www.quantamagazine.org/the-mathematics-behind-gerrymandering-20170404/.

Moon Duchin has conducted the first workshop on geometry and redistricting, at Tufts University, designed to equip participants to be expert witnesses in court cases involving redistricting. Slides and a dozen videos of presentations are available from the MGGG Website and YouTube. Regional workshops have been held in Wisconsin and North Carolina, and future ones are scheduled in Texas and California. (The Wisconsin session took place in October just after the U.S. Supreme Court heard arguments about a case involving gerrymandering in Wisconsin; this issue of MAGAZINE went to press before then.) Other features at the MGGG Website include tutorials on compactness and an open-source geographic information system for displaying compactness data of U.S. congressional districts. Klarreich's article gives examples of gerrymandering and discusses a measure of it, the "efficiency gap," which played a key role in the Wisconsin case.

McRobie, Allan, *The Seduction of Curves: The Lines of Beauty that Connect Mathematics, Art, and the Nude* (with photography by Helena Weightman), Princeton University Press, 2017; vii + 159 pp, $35. ISBN 978-0-691-17533-1.

"There is... a periodic table of curved shapes... fold, cusp, swallowtail, butterfly...." The author relies on catastrophe theory; and those four curves are the cuspoids, which with the umbilics (elliptic, parabolic, hyperbolic) are the seven elementary catastrophes. Each "can also represent a way by which something can suddenly change. This is highly relevant in engineering." Author McRobie is a professor of structural engineering who "for too many positive reasons to list here" (oh, what a copout!) introduced life drawing classes into his engineering department. And realizing the connection between the catastrophe curves and the curves of the human body led to this book, richly illustrated with diagrams, reproductions of works of art, and artistic photos of portions of the human body. The final chapters brief biography of René Thom (1923–2002), who founded catastrophe theory (but also took it far afield), and Salvador Dalí (1904–1989), who made 20 works on the theme of catastrophe theory ("Thom's Theory has bewitched all of my atoms"). As author McRobie relates, "It is not difficult to see how the two would get along so well. Dalí had spent much of his life painting distorted bodies, and Thom was the grand master of rubber-sheet geometry." On only one single page do any equations (for chemical pathway reactions) appear, inconsequentially.

Erwig, Martin, *Once Upon an Algorithm: How Stories Explain Computing*, MIT Press, 2017, xii + 319 pp, $27.95. ISBN 978-0-262-03663-4.

Recent articles and letters in *Communications of the Association for Computing Machinery* show computer scientists once again racking their brains over what "computational thinking" is. Author Erwig asserts that computation solves problems; but since there is also problem solving that is a eureka moment or otherwise unsystematic (hence noncomputational), computation requires further delineation: It is algorithm execution. What is remarkable about this book is that it illustrates how consideration of familiar stories (Hansel and Gretel, Sherlock Holmes, Indiana Jones, Over the Rainbow, Groundhog Day, Back to the Future, and Harry Potter) can bring out features about algorithms, data structures, control structures, recursion, and abstract data types. Virtually all of the concepts of a first-year course in computer science (algorithms plus data structures) are introduced, discussed, and illuminated to an unusual degree—with the use of very little code. The Introduction of the book is notable for setting out what each chapter takes on and being explicit and down-to-earth about why the ideas in it matter.

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John Watkins, *Colorado College*
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